# Tilings by translation: enumeration by a rational language approach

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**Abstract**

Beauquier and Nivat introduced and gave a characterization of the class of *pseudo-square* polyominoes that tile the plane by translation: a polyomino tiles the plane by translation if and only if its boundary word $W$ may be factorized as $W = XY\overline{X}\,\overline{Y}$. In this paper we consider the subclass *PSP* of pseudo-square polyominoes which are also *parallelogram*. By using the Beauquier-Nivat characterization we provide by means of a rational language the enumeration of the subclass of *psp*-polyominoes with a fixed planar basis according to the semi-perimeter. The case of pseudo-square convex polyominoes is also analyzed.

## 1 Introduction

The way of tiling planar surfaces has always been a fascinating problem, and it has been widely studied also in ancient time for its beautiful decorative implications.

Recently this problem has shown interesting mathematical aspects connected with computational theory, mathematical logic and discrete geometry, and tilings are often regarded as basic objects for proving undecidability results for planar problems. Furthermore, they have been used in physics, as powerful tools for studying quasi-crystal structures: in particular these structures can be better understood by representing them as rigid tilings decorated by atoms in a uniform fashion. Their long-range order can successively be investigated in a purely tiling framework, after assigning to every tiling a structural energy.

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It seems that a so wide usage of tilings (also in different disciplines) can be imputed to their capability to generate very complex configurations. These words find a confirmation in a classical result of Berger [2]: given a set of tiles, it is not decidable whether there exists a tiling of the plane which involves all its elements. This result has been achieved by constructing an aperiodic set of tiles, and successively it has been strengthened by Gurevich and Koriakov [11] to the periodic case.

Further interesting results have been achieved by restricting the class of sets of tiles only to those having one single element. In particular Wijshoff and Van Leeuwen [24] considered the *exact polyominoes* (i.e. polyominoes which tile the plane by translation) and proved that the problem of recognizing them is decidable. In [8], Beauquier and Nivat studied the same problem from a purely geometrical point of view and they found a characterization of all the exact polyominoes by using properties of the words which describe their boundaries. In particular they stated that the boundary word coding these polyominoes shows a pattern $XYZ\overline{X}\,\overline{Y}\,\overline{Z}$, called a *pseudo hexagon*, where one of the variable may be empty in which case the pattern $XY\overline{X}\,\overline{Y}$ is called a *pseudo-square*. However, in their work, the authors do not study the combinatorial properties of these structures.

Invented by Golomb [10] who coined the term *polyomino*, these well-known combinatorial objects are related to many challenging problems, such as tilings [8, 9], games [7] among many others.

The enumeration problem for general polyominoes is difficult to solve and still open. The number $a_n$ of polyominoes with $n$ cells is known up to $n = 56$ [14] and the asymptotic behavior of the sequence $\{a_n\}_{n\geq 0}$ is partially known by the relation $lim_n \{a_n\}^{1/n} = \mu$, $3.96 < \mu < 4.64$, where the lower bound is a recent improvement [1]. Nevertheless, several subclasses were enumerated by putting on polyominoes constraints. For instance, it is known [17, 22] that the number of parallelogram polyominoes having semi-perimeter $n + 1$ is the $n$-th *Catalan number* (sequence M1459 in [21]),

$$\frac{1}{n+1}\binom{2n}{n}.$$

We refer the reader to the surveys [23, 3] for the exact enumeration of various classes of polyominoes.

In this paper we study the class of convex polyominoes that also tile the plane by translation.

First we consider pseudo-square parallelogram polyominoes, and in this case it turns out that, by constraining the bottom (i.e. the component $Y$ in the decomposition $XY\overline{X}\,\overline{Y}$) to be fixed, these *psp*-polyominoes are described by a rational language, whose enumeration is straightforward.

Then we study the case of pseudo-square convex polyominoes which are not parallelogram. In this class, we can prove that a polyomino has either a unique pseudo-square decomposition and then an easy enumeration by a rational generating function, or two decompositions and then an enumeration by an infinite summation of rational generating functions.

While the convexity constrain leads to algebraic generating functions [3], it seems that the property of being pseudo-square, which is a "global" property of the boundary, gives some more complex kind of generating functions. Since we have not been able to determine an explicit expression for them, we investigate their nature according to a hierarchy which has been formalized in some recent works (see [12, 18]). The generating functions of the most common solved models in mathematical physics are *differentiably finite* (or *D-finite*), and such functions have a rather simple behavior (for instance, the coefficients can be computed quickly in a simple way; they have a nice asymptotic expansion; they can be handled using computer algebra). On the contrary, models leading to non D-finite functions are usually considered "unsolvable".

Recently many authors have applied different techniques to prove the non D-finiteness of models arising from physics or statistics [4, 5, 18, 19, 20]. By the way, T. Guttmann and I. Enting [12, 13] developed a numerical method for testing the "solvability" of lattice models, based on the study of the singularities of their *anisotropic generating functions*. Concerning the case of pseudo-squares, the test helps us to formulate the conjecture that the generating functions of the studied classes are not differentiably finite.

## 2 Pseudo-square parallelogram polyominoes

In the plane $\mathbb{Z} \times \mathbb{Z}$ a *cell* is a unit square, and a *polyomino* is a finite connected union of cells having no cut point (see Figure 1). Polyominoes are defined up to translations. A

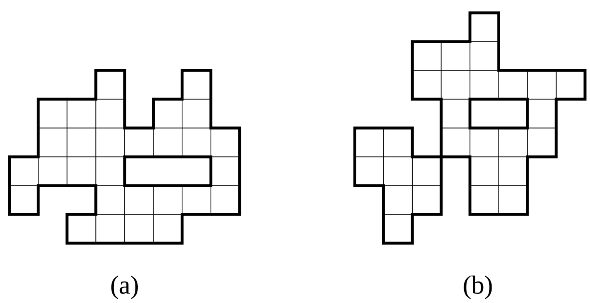


Figure 1: A polyomino (a) and a non polyomino (b).

*column* (*row*) of a polyomino is the intersection between the polyomino and an infinite strip of cells whose centers lie on a vertical (horizontal) line. A polyomino is said to be *column-convex* (resp. *row-convex*) when its intersection with any vertical (resp. horizontal) line is convex. A polyomino is *convex* if it is both column and row convex (Figure 2). In a convex polyomino, the *perimeter* is the length of its boundary and the *area* is the number of its cells. Note that the semi-perimeter is equal to the sum of the numbers of its rows and columns.

A particular subclass of the class of convex polyominoes consists of the *parallelogram* polyominoes, defined by two lattice paths that use north (vertical) and east (horizontal)

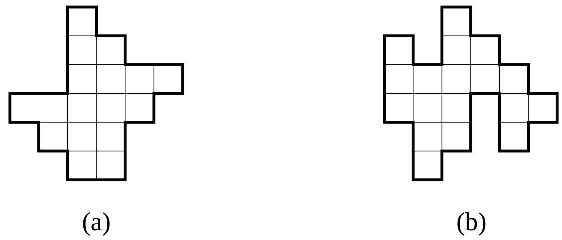



Figure 2: $(a)$ convex polyomino; $(b)$ a column-convex polyomino.

unitary steps, and intersect only at their origin and extremity. These paths are commonly called the *upper* and the *lower path*. Without loss of generality we assume that the upper and lower path of the polyomino start in $(0,0)$. Figure 3 depicts a parallelogram polyomino having area 14 and semi-perimeter 10. The boundary of a parallelogram polyomino is

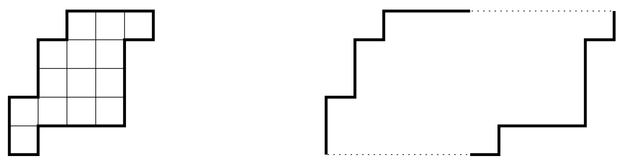

Figure 3: A parallelogram polyomino, its upper and lower paths.

conveniently represented by a *boundary word* defined on the alphabet $\{0,1\}$, where 0 and 1 stand for the horizontal and vertical step, respectively. The coding follows the boundary of the polyomino starting from $(0,0)$ in a clockwise orientation. For instance, the polyomino in Figure 3 is represented by the word

$$11011010001011100010.$$

Borrowing from [15] the basic terminology on words, if $X = u_1 \ldots u_k$ is a binary word, we indicate by $\overline{X}$ the mirror image of $X$, i.e. the word $u_k \ldots u_1$, and the length of $X$ is $|X| = k$. Moreover $|Y|_0$, (resp. $|Y|_1$) indicates the number of occurrences of 0s (resp. 1s) in $Y$.

Beauquier and Nivat [8] introduced the class of *pseudo-square* polyominoes, and prove that each polyomino of this class may be used to tile the plane by translation. Indeed, let $A$ and $B$ be two discrete points on the boundary of a polyomino $P$. Then $[A,B]$ and $\overline{[A,B]}$) denote respectively the paths from A to B on the boundary of $P$ traversed in a clockwise and counterclockwise way. The point $A'$ is the opposite of $A$ on the boundary of $P$ and s satisfies $|[A,A']| = |[A',A]|$. A polyomino $P$ is said to be *pseudo-square* if there are four points A,B,A',B' on its boundary such that $B \in [A,A']$, $[A,B] = \overline{[B',A']}$, and $[B,A'] = \overline{[A,B']}$ (see Figure 4).

In this paper we tackle the problem of enumerating pseudo-square convex polyominoes according to the semi-perimeter.

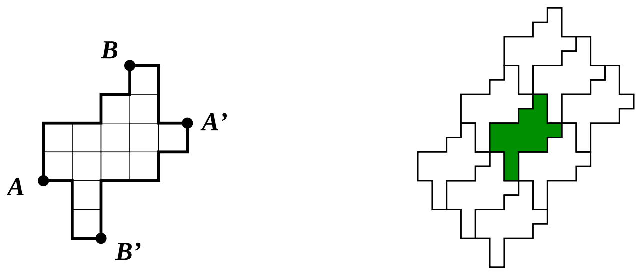


Figure 4: A pseudo-square polyomino, its decomposition and a tiling.

## 3 Pseudo-square parallelogram polyominoes

In this section we consider the class $\mathcal{PSP}$ of parallelogram polyominoes which are also pseudo-square (briefly, *psp*-polyominoes). The following properties of the class of *psp*-polyominoes are useful.

**Proposition 3.1** *If $X\,Y\,\overline{X}\,\overline{Y}$, is a decomposition of the boundary word of a psp-polyomino then $XY$ encodes its upper path, and $YX$ its lower path.*

**Proof.** The boundary word of $P$ is decomposed as $X\,Y\,\overline{X}\,\overline{Y}$. By definition of pseudo-square polyomino, we can identified $[A,B] = \overline{X}$ and $[B,A'] = Y$. Thus we find $X = [A,B] = \overline{[B',A']} = \overline{X}$ and $Y = [B,A'] = \overline{[A,B']} = \overline{Y}$. The upper and the lower paths can be written by concatenation of paths and using that $\overline{\overline{Z}} = Z$ as $U = [A,A'] = [A,B].[B,A'] = XY$ and $L = [A,A'] = [A,B'].[B',A'] = YX$. $\square$

**Proposition 3.2** *Let $P$ be psp-polyomino, whose boundary word is decomposed as $X\,Y\,\overline{X}\,\overline{Y}$. It holds that $X$ starts and ends with a $1$, and $Y$ starts and ends with a $0$.*

**Proof.** By the last proposition $P$ is decomposed in $U = XY$ and $L = YX$. As $P$ is a parallelogram polyomino the starting point is $(0,0)$ and the paths are only constituted with east and north steps. Thus the upper path begins by 1 and $U = XY$ implies that $X = 1X'$. The lower path begins by 0 and $L = YX$ implies that $Y = 0Y'$. The same reasoning applied to the ending point gives that $Y = Y''0$ and $X = X''1$. To summarize, $X$ begins and ends with a 1, and $Y$ begins and ends with a 0. $\square$

**Proposition 3.3** *A polyomino is a psp-polyomino if and only if its boundary word has unique decomposition as $X\,Y\,\overline{X}\,\overline{Y}$.*

**Proof.** We only have to prove that a *psp*-polyomino has a unique decomposition. Let us proceed by contradiction. Suppose that the boundary of $P$ has at least two decompositions. Thus the upper path is $U = XY = X'Y'$ and the lower path is $L = YX = Y'X'$. Without loss of generality, we can consider that the length of $X$ is lower than the length of $X'$ ($|X| < |X'|$) this implies that $|Y'| < |Y|$. We introduce $M$ to be the common part of $X'$ and $Y$ thus $U = XY = X'Y' = XMY'$ with $X' = XM$ and $Y = MY'$. Now the lower path can be written as $L = YX = MY'X = Y'X' = Y'XM$. We pose $W = Y'X$ and then we find $MW = WM$. By a classical lemma of combinatorics on words (see [15]) it exists $w$ a finite word and $k, \ell$ two non zero integers such that $M = w^k$ and $M = w^\ell$. Using these equations on words, the lower path is now periodic $L = MY'X = w^{k+\ell}$ and the upper path is also periodic as $U = XMY'$ is a conjugate (circular permutation of letters) of $L$ and we find $L = w'^{k+l}$. As $w$ and $w'$ are conjugated and $|w| = |w'|$ are the period then $|w|_0 = |w'|_0$ and $|w|_1 = |w'|_1$. In conclusion the point $(|w|_0, |w|_1)$ is a common point of the upper and the lower paths strictly between the starting point and the ending point of the parallelogram, in contradiction with the fact that $P$ is a polyomino. $\square$

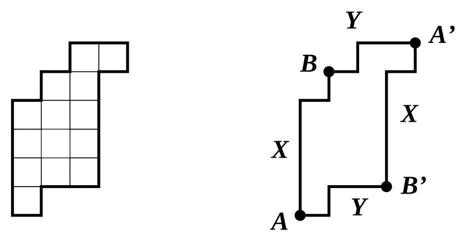


Figure 5: A *psp*-polyomino, and its unique decomposition.

For instance, the polyomino in Figure 5 can be decomposed as

$$W = 111101 \cdot 0100 \cdot 101111 \cdot 0010$$

where $X = 111101$, $Y = 0100$. We remark that the statement of Proposition 3.3 does not prevent the existence of *psp*-polyominoes having the same upper path, as shown in Fig. 6.

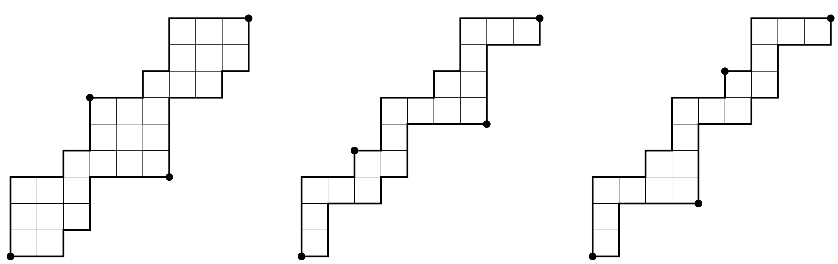

Figure 6: Three *psp*-polyominoes having the same upper path.

## 3.1 *psp*-polyominoes with flat bottom

We consider now the *psp*-polyominoes *with flat bottom*, i.e. those polyominoes such that the word $Y$ (called the *bottom*) is made only of zeroes (see Figure 7). In this section the enumeration problem for this class is solved, while the next section shows the case of *psp*-polyominoes with a generic bottom.

Let us denote by $\mathcal{PSP}^{-}$ the class of these polyominoes, and by $\mathcal{PSP}^{\overline{k}}$ the ones that have a bottom of length $k \geq 1$. If $P$ is a polyomino in $\mathcal{PSP}^{\overline{k}}$, then the word representing the upper path is:

$$X\,Y = 1\,X'\,1\,0^k,$$

for some $X'$. The following immediate property characterizes the elements of $\mathcal{PSP}^{\overline{k}}$.

**Proposition 3.4** *The word $U = 1\,X'\,1\,0^k$, with $k \geq 1$ represents the upper path of a polyomino in $\mathcal{PSP}^{\overline{k}}$ if and only if $X'$ does not contain any factor $0^j$, with $j \geq k$.*

**Proof.**
$(\Rightarrow)$ Suppose by contradiction that $U = 1\,X'\,1\,0^k$ encodes the upper path of a parallelogram polyomino $P$, and $X'$ contains a factor $0^k$, so that we can write $U$ as

$$U \;=\; 1\,X''0^kX'''\,1\,0^k, \qquad X'',\, X''' \in \{0,1\}^*.$$

The lower path of $P$ can thus be encoded as

$$L \;=\; 0^k\,1\,X''0^kX'''\,1.$$

It follows that the upper and lower path meet in $(\,k + |X''|_0\;,\; 1 + |X''|_1\,)$, so $P$ is not a polyomino, which contradicts our initial hypothesis.

$(\Leftarrow)$It can be proved in a completely analogous way. $\square$

**Example 3.1** The word 11001000111010011000 1 represents the upper path of a polyomino in $\mathcal{PSP}^{\overline{4}}$, as shown in Figure 7 (a), while the word 101100000101 does not encode a polyomino since it contains the factor 00000 (Figure 7 (b)).

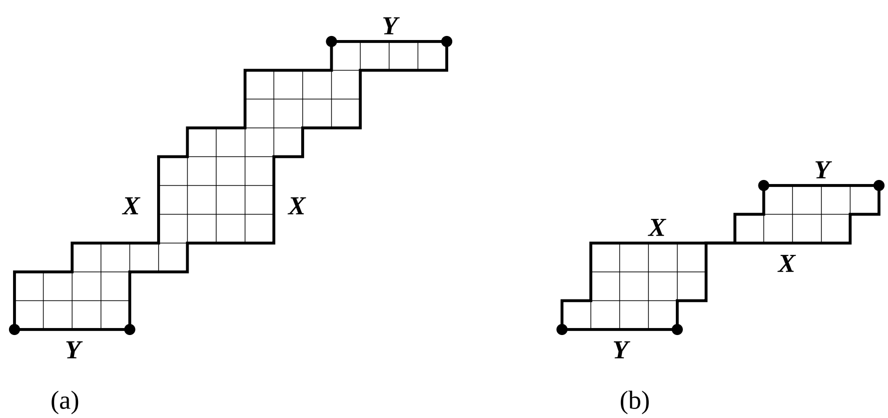


Figure 7: Examples

| $\overline{q}_n$ | $k=1$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | | |
| 2 | 1 | 1 | | | | | | | |
| 3 | 1 | 1 | 1 | | | | | | |
| 5 | 1 | 2 | 1 | 1 | | | | | |
| 8 | 1 | 3 | 2 | 1 | 1 | | | | |
| 14 | 1 | 5 | 4 | 2 | 1 | 1 | | | |
| 24 | 1 | 8 | 7 | 4 | 2 | 1 | 1 | | |
| 43 | 1 | 13 | 13 | 8 | 4 | 2 | 1 | 1 | |
| 77 | 1 | 21 | 24 | 15 | 8 | 4 | 2 | 1 | 1 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

Table 1: the number $p_n^{\overline{k}}$ of *psp*-polyominoes with flat bottom of length $k$ .

In Table 1 are displayed the numbers $p_n^{\overline{k}}$ of *psp*-polyominoes with flat bottom of length $k$ having semi-perimeter equal to $n$, for $k = 1, \ldots, 9$, $n \geq 2$.

Clearly, the number $p_n^-$ of *psp*-polyominoes of $\mathcal{PSP}^-$ having semi-perimeter equal to $n$, reported in the first column of Table 1, is given by the sum:

$$p_n^- = \sum_{k \geq 1} p_n^{\overline{k}}.$$

Using the result in Proposition 3.4 we observe that each word $W$ can be uniquely decomposed as:

$$W = \, 1\, p_1 \ldots p_s \, 0^k,$$

where,

$$p_j \in \left(1 \cup 01 \cup 001 \cup \ldots \cup 0^{k-1}1\right), \qquad j = 1, \ldots, s, \tag{1}$$

thus $W$ is a word of the regular language defined by the expression:

$$1\left(1 \cup 01 \cup 001 \cup \ldots \cup 0^{k-1}1\right)^* 0^k.$$

For example, the word representing the upper path of the polyomino in $\mathcal{PSP}^{\overline{4}}$ depicted in Figure 7, (a) has a unique decomposition as

$$1\ 1\ 001\ 0001\ 1\ 1\ 01\ 001\ 1\ 0001\ 0000.$$

Translating this argument into generating functions, we have that, for any fixed $k \geq 1$ the generating function of the class $\mathcal{PSP}^{\overline{k}}$:

$$f_k(x) = \frac{x^{k+1}}{1 - x - x^2 - x^3 - \ldots - x^k}. \tag{2}$$

Finally, the generating function of the class $\mathcal{PSP}^-$ is given by the sum:

$$f(x) = \sum_{k\geq 1} f_k(x) = x(1-x)\sum_{i\geq 1}\frac{x^i}{1-2x+x^{i+1}} = x^2+2x^3+3x^4+5x^5+8x^6+14x^7+24x^7+\ldots, \tag{3}$$

defining the sequence A079500 in [21].

In [16] A. Knopfmacher and N. Robbins prove that the coefficient $f'_n$ of $f(x)/x$ is the number of *compositions* of the integer $n$ for which *the largest summand occurs in the first position*, and that, as $n \to \infty$

$$f'_n \sim \frac{2^n}{n\log 2}(1+\delta(\log_2 n)),$$

where $\delta(x)$ is a continuous periodic function of period 1, mean zero, and small amplitude. In the same paper it is showed the nice property that the coefficient $f'_n$ is odd if and only if $n = m^2 - 1$, or $n = m^2$, with $m \geq 1$.

We are not able to find a closed expression for $f(x)$, free from summation symbols, but we are able to state something about its nature. In [6], page 298, P. Flajolet studies the function:

$$\frac{x(1-x)}{1-2x}\sum_{i\geq 0}\frac{x^{2i}}{1-2x+x^{i+1}}, \tag{4}$$

and in particular he proves that it is not *differentiably finite*. We recall that a formal power series in $u(x)$ with coefficients in $\mathbb{C}$ is said to be *differentiably finite* (briefly, *D-finite*) if it satisfies a (non-trivial) polynomial equation:

$$q_m(x)u^{(m)} + q_{m-1}(x)u^{(m-1)} + \ldots + q_1(x)u' + q_0(x)u = q(x),$$

with $q_0(x), \ldots, q_m(x) \in \mathbb{C}[x]$, and $q_m(x) \neq 0$ ([22], Ch.6).

Flajolet's proof bases on the very simple argument, arising from the classical theory of linear differential equations, that a D-finite power series of a single variable has only a finite number of singularities. Thus non D-finiteness follows from the proof that the function has infinitely many zeros.

The same reasoning can be applied in order to state that the generating function $f(x)$ of *psp*-polyominoes with flat bottom is not D-finite.

## 3.2 Enumeration of *psp*-polyominoes with fixed bottom

In this section we consider the enumeration of *psp*-polyominoes with a generic fixed bottom $Y = 0Y'0$, $Y' \in \{0,1\}^*$. We say that a binary word $X$ is *compatible* with $Y$ if the word $X\,Y\,\overline{X}\,\overline{Y}$ represents the boundary of a *psp*-polyomino. The problem is now to give a characterization of the words which are compatible with $Y$. We reach this goal by determining the regular language $\mathcal{L}_Y$ of words $XY$ such that $X$ is compatible with $Y$.

Let us start by giving some definitions. Let $\mathcal{F}(Y)$ (briefly $\mathcal{F}$) be the (finite) set

$$\mathcal{F} \;=\; \{\, W \in \{0,1\}^* \;:\; |W| = |Y| \;\wedge\; |W|_0 \geq |Y|_0 \,\}\,,$$

and, let $\mathcal{L}_F$ be the regular language consisting of all the words that do not contain any element of $\mathcal{F}$ as factor:

$$\mathcal{L}_F \;=\; \{0,1\}^* \setminus \{0,1\}^* \,\mathcal{F}\, \{0,1\}^*.$$

Moreover, let us consider the (finite) set of paths starting from $(0,0)$, ending to the line $y = |Y|_1 + 1$, using north and east unitary steps and never touching the path defined by the bottom $Y$, and let $\mathcal{I}$ be the set of words encoding these paths. Roughly speaking, the words in $\mathcal{I}$ are all the possible prefixes for $XY$, being $X$ compatible with $Y$. The words of $\mathcal{I}$ can be determined graphically, as shown in the next example.

**Example 3.2** Given the bottom $Y = 001010$, we have that $\mathcal{F}$ is made of all the binary words of length 6 having more than three 0's, and $\mathcal{I} = \{111, 1101, 1011, 11001, 10101\}$ (see Figure 8).

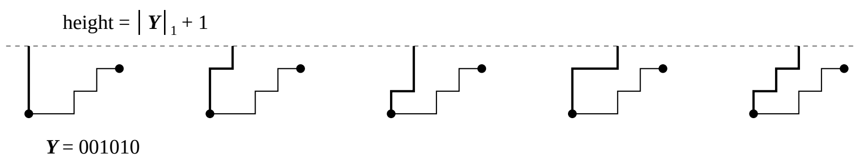


Figure 8: The initial language $\mathcal{I}$.

Now we have set all the definitions necessary to construct the (regular) language:

$$\mathcal{L}_Y \;=\; (\; \mathcal{I}\,\{0,1\}^* \;\cap\; \{0,1\}^* 0 Y' \;\cap\; \mathcal{L}_F \;) \cdot 0.$$

**Proposition 3.5** *A binary word $XY$ represents the upper path of a* psp-*polyomino with bottom $Y$ if and only if $XY \in \mathcal{L}_Y$.*

**Proof.** $(\Rightarrow)$ Let $XY$ represent the upper path of a *psp*-polyomino $P$ with bottom $Y$. We want to prove that $XY \in \mathcal{L}_Y$. Since it can be easily checked that $XY$ begins with a word in $\mathcal{I}$, and ends with $0Y'0 = Y$, it remains only to show that $XY \in \mathcal{L}_F 0$, i.e. $X0Y' \in \mathcal{L}_F$.

Let us assume, by contradiction, that $X0Y' \notin \mathcal{L}_F$, i.e. there is at least a factor $Z$ of $X0Y'$, such that $|Z| = |Y|$, and $|Z|_0 = |Y|_0$. Accordingly, the boundary word encoding the upper path of $P$ may be decomposed as:

$$X\,Y \;=\; S\,Z\,T\,0, \qquad \text{with } S, T \in \{0,1\}^*.$$

Naturally, $Z$ cannot be a factor of $Y$, since they have the same length, thus we must have:

$$X = S\,Z_X, \quad Y = Z_Y\,T\,0, \quad Z = X \cup Y, \qquad \text{with } Z_X \neq \emptyset.$$

Thus the lower path can be represented by $Y\,X = Z_Y\,T\,0\,S\,Z_X$. Now we observe that the paths encoded by $S\,Z_X\,Z_Y = S\,Z$ (which is a proper prefix of the upper path), and by $Z_Y\,T\,0\,S = Y\,S$ (which is a proper prefix of the lower path) meet at their end point, since they have the same length and the same number of 0's by hypothesis. This means that the upper and the lower path just meet before their endpoints, and it is a contradiction.

$(\Leftarrow)$ It can be proved in a completely analogous way. $\square$

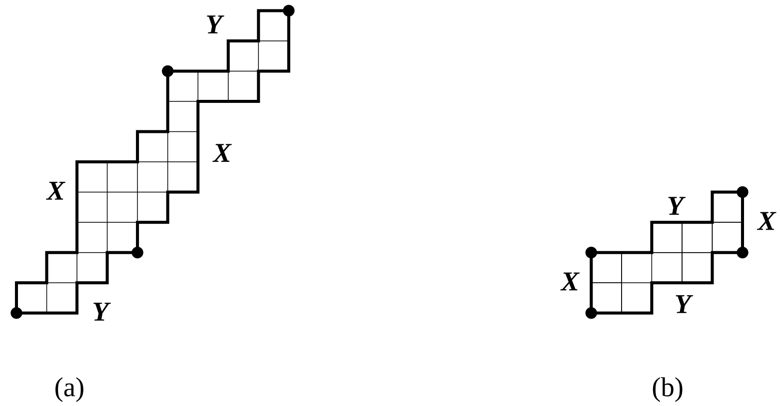


Figure 9: (a) The polyomino of Example 3.3; (b) A polyomino where the initial factor $I$ overlaps $Y$: $X = 11$, $Y = 0010010$, and $I = 11001$.

**Example 3.3** Referring to Example 3.2, let us consider the *psp*-polyomino shown in Figure 9 (a), with bottom $Y = 001010$. We observe that the word representing its upper path is an element of $\mathcal{L}_Y$, since it can be decomposed as

$$10101 \cdot 11001011 \cdot 00101 \cdot 0,$$

and $101011100101100101 \in \mathcal{L}_F$, $10101 \in \mathcal{I}$, $00101 = 0Y'$.

**Remark.** Note that, based on the definition of $\mathcal{L}_Y$, a word $W = XY \in \mathcal{L}_Y$ may be decomposed also as $W = I \cdot E$, with $I \in \mathcal{I}$, and $E \in \{0,1\}^*$, thus the factor $I$ may overlap $Y$, as shown in Figure 9 (b), where we have $XY = 11{\cdot}0010010$, and $I = 11001$.

Thanks to the result of Proposition 3.5, one can easily build the automaton associated with the regular language $\mathcal{L}_Y$. Then it is easy to obtain the generating function for the class of *psp*-polyominoes having bottom $Y$, by applying the Schützenberger methodology to the automaton associated with $\mathcal{L}_Y$. A final significative example is now provided.

**Example 3.4** We determine the generating function of the set of *psp*-polyominoes having bottom $Y = 0010$ according to the semi-perimeter. The sets $\mathcal{F}$ and $I$ turn to be

$$\mathcal{F} = \{\, 0000, 1000, 0100, 0010, 0001 \,\}, \qquad \text{and} \qquad \mathcal{I} = \{\, 11, 101 \,\}.$$

From Proposition 3.5 we obtain the language:

$$\mathcal{L}_Y = (\{11, 101\} \cdot \{0,1\}^* \cap \{0,1\}^* \setminus \{0,1\}^* \cdot \mathcal{F} \cdot \{0,1\}^* \cap \{0,1\}^* \, 001\,) \cdot 0.$$

A deterministic and minimal automaton recognizing $\mathcal{L}_Y$ can easily be built, see for instance that depicted in Figure 10. On the left of the dashed vertical line are placed the initial states, necessary to impose the all the words of the language begin with 11 or 101. For sake of simplicity, the states on the right of the vertical line have been labelled with a

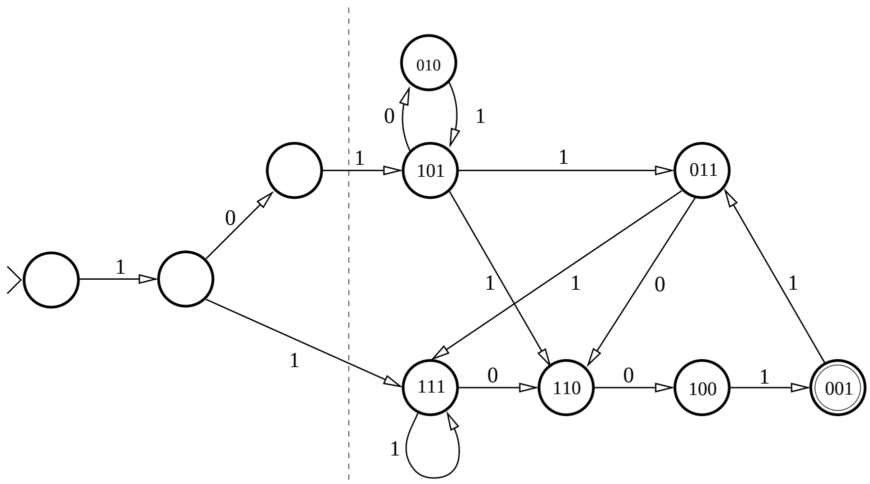


Figure 10: The automaton recognizing the language $\mathcal{L}_Y$ of Example 3.4.

word of length three (having at least one 1); each label on a state indicates the last three letters of the word that is examined when the state is reached (with the only exception of the state 111 which can initially be reached when examining the word 11). Thus we have:

$$\binom{3}{0} + \binom{3}{1} + \binom{3}{2} = 7$$

labelled states. The strong component of the automaton is nothing but the DeBruijn graph of factors of length three having at least one 1. Passing to the system of functional equations associated with the automaton, we finally calculate the generating function of the language $\mathcal{L}_Y$, i.e.

$$f_Y = \frac{x^5}{1 - x - x^2 - x^4 + x^6}\,.$$

We remark that the denominator of the generating function is completely determined by the number of 1's and 0's in $Y$, and not by their positions; for instance, the generating function of $\mathcal{L}_{0100}$

$$\frac{x^5(1-x^2)}{1-x-x^2-x^4+x^6},$$

has the same denominator as that determined in Example 3.4. This simple observation suggests the problem of determining a general expression for $Q(r,s)$, i.e. the denominator of the generating function associated with any bottom $Y$ having $r$ 0's and $l$ 1's ($r \geq 2$, $l \geq 1$). Below we give some partial results:

- $Q(2,s) = 1-x-x^3$, for any $s \geq 1$;
- $Q(3,1) = 1-x-x^2-x^4+x^6$;
- $Q(3,2) = 1-x-x^3-2x^5+x^8+x^{10}$;
- $Q(4,1) = 1-2x-2x^3+x^7+x^8$.

## 3.3 On the generating function of *psp*-polyominoes

By the results in the previous section, we have that the generating function $q(x)$ of *psp*-polyominoes according to the semi-perimeter can be obtained as the sum of the (rational) generating functions associated with all possible bottoms $Y$, i.e.

$$q(x) = \sum_{Y \in 0\{0,1\}^*0} f_Y(x),$$

At the moment we have not been able to determine a closed formula for this expression. Using a computer program we have determined the first terms of the sequence $\{q_n\}_{n\geq 2}$ defined by $q(x)$ (not in [21]) :

$1, 2, 3, 6, 11, 22, 45, 90, 184, 370, 751, 1516, 3053, 6172, 12405, 25042, 50323, 101424,$
$203880, 410296, 824871, 1658338, 3333405, 6696814, 13457112, 27021758, 54278993, \ldots$

and Figure 11 depicts the 11 *psp*-polyominoes having semi-perimeter equal to 7. Moreover, Table 2 reports the numbers of *psp*-polyominoes having semi-perimeter $n \geq 2$ and $k \geq 1$ rows.

Recently, Tony Guttmann [12] suggested a numerical procedure for testing the solvability of lattice models based on the study of the singularities of their *anisotropic generating functions*. In practice, we consider the anisotropic generating function $q(x,y)$ of *psp*-polyominoes by counting polyominoes according to the number of rows and columns,

$$q(x,y) = \sum_{m,n} q_{m,n} x^m y^n,$$

where $q_{m,n}$ is the number of *psp*-polyominoes with $m$ rows and $n$ columns. Hence we may rewrite the generating functions as:

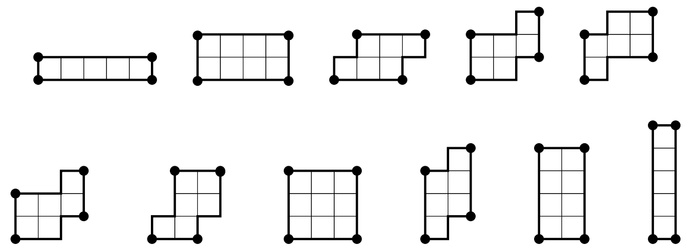

Figure 11: The 11 *psp*-polyominoes having semi-perimeter equal to 7.

| $q_n \backslash k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | | |
| 2 | 1 | 1 | | | | | | | |
| 3 | 1 | 1 | 1 | | | | | | |
| 6 | 1 | 2 | 2 | 1 | | | | | |
| 11 | 1 | 2 | 5 | 2 | 1 | | | | |
| 22 | 1 | 3 | 7 | 7 | 3 | 1 | | | |
| 45 | 1 | 3 | 11 | 15 | 11 | 3 | 1 | | |
| 90 | 1 | 4 | 15 | 25 | 25 | 15 | 4 | 1 | |
| 184 | 1 | 4 | 20 | 41 | 52 | 41 | 20 | 4 | 1 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

Table 2: The numbers of *psp*-polyominoes of $\mathcal{PSP}$ having $k$ columns, $k = 1, \ldots, 9$.

$$q(x,y) = \sum_{n \geq 1} \left( \sum_{m \geq 1} q_{m,n} x^m \right) y^n = \sum_{n \geq 1} H_n(x) y^n.$$

The series $q(x, y)$ is said to be *differentiably finite* (*D-finite*) if there is a (non-trivial) differential equation:

$$p_m(x,y) \frac{\partial^m}{\partial y^m} u(x,y) + \ldots + p_1(x,y) \frac{\partial}{\partial y} u(x,y) + p_0(x,y) u(x,y) = 0,$$

with $p_j$ a polynomial in $x$ and $y$, with complex coefficients. Guttmann's test of solvability aims at arguing whether the function $q(x, y)$ is or not D-finite, and essentially bases on the observation of first values of $H_n(x)$. Concerning our series $q(x, y)$ of *psp*-polyominoes we have:

1. $H_n(x)$ is a rational function;
2. the degree of the numerator of $H_n(x)$ is smaller than the degree of the denominator;

3. the first terms of the denominators of $H_n(x)$ (denoted by $D_n(x)$) are product of cyclotomic polynomials[1], and that, the $n$th cyclotomic polynomial appears for the first time in the term $D_n(x)$.:

$$\begin{aligned}
D_1(x) &= (1-x)\\
D_2(x) &= (1-x)^2(1+x)\\
D_3(x) &= (1-x)^3(1+x)(1+x+x^2)\\
D_4(x) &= (1-x)^4(1+x)^2(1+x+x^2)(1+x^2)\\
D_5(x) &= (1-x)^5(1+x)^3(1+x+x^2)^2(1+x^2)(1+x+x^2+x^3+x^4)\\
D_6(x) &= (1-x)^6(1+x)^3(1+x+x^2)^2(1+x^2)(1+x+x^2+x^3+x^4)(1-x+x^2).
\end{aligned}$$

T. Guttmann observed that for a large number of *unsolved models* (leading to non D-finite generating functions) the number of different factors in the denominators increases with $n$, as it happens for *psp*-polyominoes, and suggested that this property could be used as a test of *solvability.* This test has been considered successively by A. Rechnitzer for conjecturing (and then proving) the non D-finiteness of self-avoiding polygons [19], of directed bond animals [20], and of bargraphs according to the site perimeter [5]. Motivated by Guttmann's test we make the following conjecture:

**Conjecture 1.** *The anisotropic generating function of psp polyominoes is not D-finite.*

How is it now possible to prove Conjecture 1? We cannot use the same criterion used for the generating function of *psp*-polyominoes with flat bottom. Indeed, while a D-finite power series of a single variable has only a finite number of singularities, there are examples of two variables series having infinitely many singularities. Then we need to use the following:

**Theorem 3.1 ([18])** *Let $f(x,y)=\sum_{n\geq 0} H_n(x)y^n$ be a D-finite series in $y$ with coefficients $H_n(x)$ that are rational functions of $x$. For $n \geq 0$ let $S_n$ be the set of poles of $H_n(x)$, and let $S=\bigcup_n S_n$. Then $S$ has only a finite number of accumulation points.*

Thus, if the set of singularities of the denominators of the anisotropic generating function has an infinite set of accumulation points, the anisotropic generating function is not D-finite. Referring to case of *psp*-polyominoes, if properties 1., 2. and 3. (which have been verified for small $n$) are proved, then we have that the singularities of $H_n(x)$ are dense on the unit circle $|x|=1$, hence, by Theorem 3.1, the series $q(x,y)$ is not D-finite. Some helpful discussion with A. Rechnitzer suggested that, while determining the exact form for the denominator $D_n(x)$ may be a very hard task, to prove Conjecture 1 it is sufficient to prove the following weaker statement:

[1] We remind that the cyclotomic polynomials are the factor of $1-x^n$, and in particular $\prod_{k|n}\Psi_k(x)$, where $\Psi_k(x)$ is the $k$th cyclotomic polynomial.

**Conjecture 2.** *$H_k(x)$ is a rational generating function, and its denominator contains a factor $\Psi_k(x)$ which does not cancel with the numerator.*

We are also convinced that for such a proof it is convenient to use *haruspicy techniques*, as those developed in [18, 19, 20].

# 4 Pseudo-square convex polyominoes.

In this section we will treat the case of pseudo-square convex polyominoes, denoted by $\mathcal{C}$. In particular we are interested in those polyominoes in $\mathcal{C}$ which are not parallelogram ones, nor are the reflection of a parallelogram polyomino with respect to the $y$-axis. So, let $\mathcal{PSP}^*$ the class of polyominoes obtained by reflecting *psp*-polyominoes with respect to the $y$-axis. Moreover, let $\mathcal{H} = \mathcal{C} - (\mathcal{PSP} \cup \mathcal{PSP}^*)$. Let $c_n$ (resp. $q_n^*$, $h_n$) denote the number of polyominoes in $\mathcal{C}$ (resp. $\mathcal{PSP}^*$, $\mathcal{H}$) having semi-perimeter equal to $n$.

First we observe that, for any $n \geq 2$, $q_n^* = q_n$. Moreover $\mathcal{PSP}_n \cap \mathcal{PSP}_n^*$ is constituted only by rectangles, hence $|\mathcal{PSP}_n \cap \mathcal{PSP}_n^*|$ is equal to the number of integer partitions of $n$ into exactly two summands, that is:

$$|\mathcal{PSP}_n \cap \mathcal{PSP}_n^*| = n - 1. \tag{5}$$

Thus, for all sizes $n \geq 2$, we have :

$$\begin{aligned} c_n &= h_n + |\mathcal{PSP}_n \cup \mathcal{PSP}_n^*| \\ &= h_n + |\mathcal{PSP}_n| \ + \ |\mathcal{PSP}_n^*| \ - \ |\mathcal{PSP}_n \cap \mathcal{PSP}_n^*| \\ &= h_n + 2q_n - (n-1). \end{aligned} \tag{6}$$

In a polyomino $P$ of $\mathcal{H}$, let us indicate, using the letters from $A$ to $H$ in a clockwise sense, the extremal points where the minimal bounding rectangle meets with $P$ (see Figure 12). We observe that under our assumptions, the paths $[B, C]$, $[D, E]$, $[B, C]$, and $[B, C]$ need not be empty.

From now on, we will often describe the boundary of a polyomino by means of a word over the alphabet $\{N, E, S, W\}$, where $N$ (resp. $E$, $S$, $W$) stands for the *north* (resp. *east*, *south*, *west*) unit step. The word representing a polyomino is obtained simply by following its boundary from a fixed *starting point* and in a clockwise sense. Moreover, if $X = x_1 x_2 \cdots x_r$ where $x_i \in \{N, E, S, W\}$ then $\overline{X} = \overline{x_r} \cdots \overline{x_2}\ \overline{x_1}$ with the property that $\overline{N} = S, \overline{S} = N, \overline{W} = E, \overline{E} = W$.

Using this notation a polyomino is a pseudo-square if its boundary word can be decomposed in $XY\overline{X}\ \overline{Y}$ where $X$ and $Y$ are non empty words on the alphabet $\{N, E, S, W\}$.

**Proposition 4.1** *If $P$ is a polyomino of $\mathcal{H}$, then it can have the following two decompositions:*

($\alpha$) *starting from A:*

$$X = [A, C] \quad Y = [C, E]$$

$$\overline{X} = [E, G] \quad \overline{Y} = [G, A].$$

($\beta$) *starting from B:*

$$X = [B, D] \quad Y = [D, F]$$

$$\overline{X} = [F, H] \quad \overline{Y} = [H, B].$$

**Proof.** Let $P$ be a polyomino of $\mathcal{H}$, and $XY\overline{X}\,\overline{Y}$ a decomposition of its boundary. We prove that the only discrete points which can be the first point of a component in a decomposition are $A$, $B$, $C$, $D$, $E$, $F$, $G$, and $H$.

We start considering the path running from $A$ to $C$ in a clockwise sense. We first observe that no point between $B$ and $C$, except $B$ and $C$ themselves, can be the first point of a component (say the component $X$, without loss of generality), due to the convexity of $P$. So let us assume by that there is a point $O$ between $A$ and $B$ (and $O \neq A, B$) which is the first point of $X$. Thus $X$ begins with an $N$ step, and $\overline{Y}$ ends with an $N$ step, which means that $Y$ begins with an $S$ step. For this reason, and because of the convexity of $P$, $X$ must end with an $E$ step, and thus $\overline{X}$ begins with an $O$ step, and ends with a $S$ one. Since $\overline{X}$ meets with $\overline{Y}$, and for convexity reasons, we have that the first step of $\overline{Y}$ must be an $S$ step. Accordingly we have that $Y$ final step is an $E$, which contradicts the fact that the first step of $\overline{X}$ is an $O$.

Analogously we prove that the other points in the boundary that can be the first points of a component in a decomposition $XY\overline{X}\,\overline{Y}$ of the boundary of $P$ are $D$, $F$, $G$, and $H$.

If the first point of $X$ is $A$, then $X$ begins with an $N$, hence $\overline{Y}$ ends with an $O$ step, and $Y$ begins with an $E$ step. Thus $X$ must end in $C$, i.e. $X = [A, C]$, and then $Y = [C, E]$. Similarly, if the first point of $X$ is $B$ we have $X = [B, D]$, and $Y = [D, F]$. □

According to Proposition 4.1 we can distinguish among three types of polyominoes of $\mathcal{H}$:

i) polyominoes which have one decomposition of type ($\alpha$), belonging to the class $\mathcal{H}^{\alpha}$ (see Fig. 12 (a));

ii) polyominoes which have one decomposition of type ($\beta$), belonging to the class $\mathcal{H}^{\beta}$ (see Fig. 12 (b));

iii) polyominoes which have two different possible decompositions, one of type ($\alpha$), and one of type ($\beta$), belonging to the class $\mathcal{H}^{\alpha} \cap \mathcal{H}^{\beta}$, denoted by $\mathcal{H}^{\alpha\wedge\beta}$ (see Fig. 13 (a),(b)).

As usual, for any $n \geq 0$, $\mathcal{H}_n$ (resp. $\mathcal{H}_n^{\alpha}$, $\mathcal{H}_n^{\beta}$, $\mathcal{H}_n^{\alpha\wedge\beta}$) denotes the set of polyominoes of $\mathcal{H}$ (resp. $\mathcal{H}^{\alpha}$, $\mathcal{H}^{\beta}$, $\mathcal{H}^{\alpha\wedge\beta}$) having semi-perimeter equal to $n$. For symmetry reasons, $|\mathcal{H}_n^{\alpha}| = \left|\mathcal{H}_n^{\beta}\right|$, thus:

$$|\mathcal{H}_n| = |\mathcal{H}_n^{\alpha}| + \left|\mathcal{H}_n^{\beta}\right| - \left|\mathcal{H}_n^{\alpha\wedge\beta}\right| = 2\,|\mathcal{H}_n^{\alpha}| - \left|\mathcal{H}_n^{\alpha\wedge\beta}\right|.$$

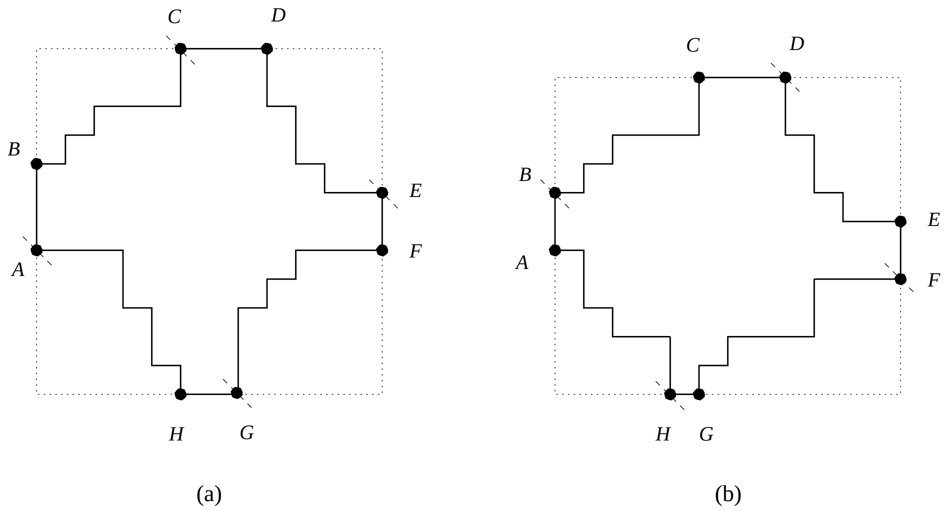


Figure 12: (a) A pseudo-square convex polyomino not parallelogram of type $\alpha$. (b) A pseudo-square convex polyomino not parallelogram of type $\beta$. Observe that the path from $B$ to $F$ is the same in the two polyominoes.

## 4.1 The generating function of $\mathcal{H}^\alpha$

Since each polyomino of $\mathcal{H}^\alpha$ is convex and pseudo-square, and its boundary has a unique decomposition such that $X = [A, C]$, and $Y = [C, E]$, it is trivial that the path $[A, B]$ uses only north unitary steps, the path $[B, C]$ uses only north and east steps, begins with an east and ends with a north one, the path $[C, D]$ uses only east steps, and the path $[D, E]$ uses only south and east steps, begins with a south step and ends with an east one. Moreover, by definition of the class $\mathcal{H}$, $[B, C]$ and $[D, E]$ cannot be empty paths, and consequently also $[A, B]$ and $[C, D]$ contain at least one step.

These properties easily lead to the solution of the enumeration problem for $\mathcal{H}^\alpha$; indeed, the generating function $h^\alpha(x)$ for the class $\mathcal{H}^\alpha$ can be obtained as the product of the generating functions for the paths $[A, B]$, $[B, C]$, $[C, D]$, and $[D, E]$:

$$h(x)^\alpha = h^\alpha_{[A,B]}(x) \cdot h^\alpha_{[B,C]}(x) \cdot h^\alpha_{[C,D]}(x) \cdot h^\alpha_{[D,E]}(x).$$

Simple combinatorial arguments now yield the computation of these generating functions:

$$h^\alpha_{[A,B]}(x) = h^\alpha_{[C,D]}(x) = \frac{x}{1-x} \qquad h^\alpha_{[B,C]}(x) = h^\alpha_{[D,E]}(x) = \frac{x^2}{1-2x},$$

and finally, we have:

$$h^\alpha(x) = \frac{x^6}{(1-x)^2(1-2x)^2}. \tag{7}$$

The first terms of the sequence $h^\alpha_n$ are $1, 6, 23, 72, 201, 522, \ldots$ (sequence A045618 in [21]). For instance Figure 14 shows the 6 polyominoes in $\mathcal{H}^\alpha$ having semi-perimeter equal to 7.

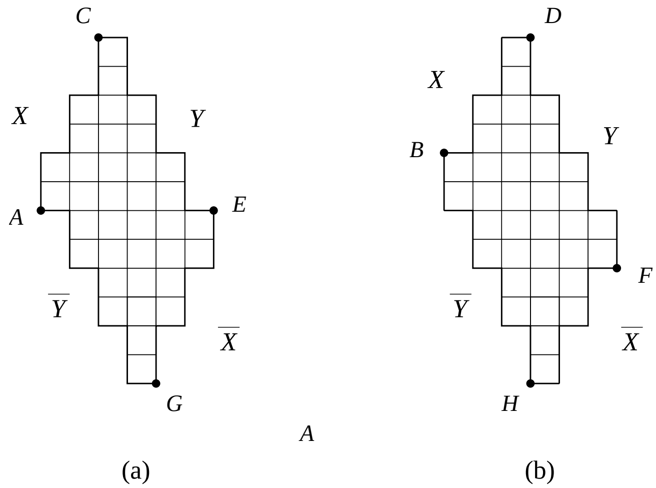


Figure 13: (a), (b) A polyomino in $\mathcal{H}_n^{\alpha\wedge\beta}$ and its two different decompositions.

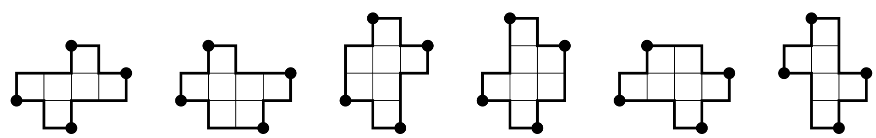

Figure 14: The 6 polyominoes in $\mathcal{H}^{\alpha}$ having semi-perimeter equal to 7.

### 4.1.1 The generating function of $\mathcal{H}^{\alpha\wedge\beta}$

We start giving a property which characterizes the polyominoes having two different decompositions:

**Proposition 4.2** *If $P$ is a polyomino of $\mathcal{H}^{\alpha\wedge\beta}$, then the two decompositions are given by:*

$$(\alpha): \quad \begin{array}{ll} X = (N^sE^r)^kN^s & Y = (E^rS^s)^{k'}E^r \\ \overline{X} = (S^sO^r)^kS^s & \overline{Y} = (O^rN^s)^{k'}O^r, \end{array}$$

$$(\beta): \quad \begin{array}{ll} X' = (E^rN^s)^kE^r & Y' = (S^sE^r)^{k'}S^s \\ \overline{X}' = (O^rS^s)^kO^r & \overline{Y}' = (N^sO^r)^{k'}N^s, \end{array}$$

*with $r, s \geq 1, k, k' \geq 1$, where, as usual, it is assumed that the decomposition $(\alpha)$ starts from the point $A$ of $P$, and the decomposition $\beta$ starts from the point $B$, and $N, O, S, E$ denote, as usual, the north, west, south, and east unitary steps, respectively.*

**Proof.** The boundary word of $P$, starting from $A$, can be written as

$$N^s\ T\ E^r\ U\ S^s\ R\, W^r\ V,$$

where $T \in \{E,N\}^*$, $U \in \{E,S\}^*$, $R \in \{S,W\}^*$ and $W \in \{W,N\}^*$. Let us assume that the boundary has two decompositions, according to Proposition 4.1, of types $(\alpha)$ and $(\beta)$:

$$X = N^s T, \quad Y = E^r U, \quad \overline{X} = S^s R, \quad \overline{Y} = W^r V,$$

$$X' = TE^r, \quad Y' = US^s, \quad \overline{X'} = RW^r, \quad \overline{Y'} = VN^s.$$

Thus $X = N^s T$ and $\overline{X} = S^s R$ implies that $X = N^s T = \overline{\overline{X}} = \overline{R} N^s$. In the same way, $X' = TE^r = E^r \overline{R}$. Then $T$ begins by $E^r$ and ends by $N^s$. We can write $T = E^r T' N^s$ and by substitution

$$X = N^s E^r T' N^s \qquad X' = E^r T' N^s E^r.$$

Using the $^-$ operator, we find

$$\overline{X} = S^s \overline{T'} W^r S^s \qquad \overline{X'} = W^r S^s \overline{T'} W^r.$$

As $\overline{X'} = RW^r$ then $R = W^r S^s \overline{T'}$ and as $X = \overline{R} N^s$ then $\overline{X} = S^s R$ and $R = \overline{T'} W^r S^s$. By these equalities,

$$R = W^r S^s \overline{T'} = \overline{T'} W^r S^s$$

and by solving this equation on words we obtain that $\overline{T'} = (W^r S^s)^k$, with $k \geq 0$. By substitution of

$$T' = \overline{\overline{T'}} = (N^s E^r)^k$$

in $T = E^r T' N^s$ we obtain that

$$T = E^r (N^s E^r)^k N^s = (E^r N^s)^{k+1}, \text{ with } k \geq 0.$$

Using the last equality, we find that

$$X = N^s (E^r N^s)^{k+1} = (N^s E^r)^{k+1} N^s \text{ with } k \geq 0.$$

Thus $X = (N^s E^r)^k N^s$ with $k \geq 1$.

The same reasoning on $Y$ and $Y'$ leads to $Y = (E^r S^s)^{k'} E^r$, and $Y' = (S^s E^r)^{k'} S^s$. $\square$

**Remark.** By Proposition 4.2, the smallest polyomino in $\mathcal{H}^{\alpha\wedge\beta}$ is obtained when $r = s = k = k' = 1$, and it is the "cross" having the two possible decompositions $NEN\ ESE\ SOS\ ONO$, and $ENE\ SES\ OSO\ NON$.

For any fixed $s, r \geq 1$, then the generating function of the polyominoes of $\mathcal{H}^{\alpha\wedge\beta}$ having $X = (N^s E^r)^k N^s$, $Y = (E^r S^s)^{k'} E^r$, according to the semi-perimeter is given by:

$$f_{r,s}(x) = \frac{x^{3(r+s)}}{(1 - x^{r+s})^2}.$$

Now to obtain the generating function of $\mathcal{H}^{\alpha\wedge\beta}$ we must sum $f_{r,s}(x)$ over all possible $r, s \geq 1$, i.e.

$$h^{\alpha\wedge\beta}(x) = \sum_{r,s\geq 1} f_{r,s}(x). \tag{8}$$

We observe that for any $r, s, r', s' \geq 1$ such that $r+s = r'+s'$ we have $f_{r,s}(x) = f_{r',s'}(x)$; moreover, the number of pairs $(r, s)$, $r, s \geq 1$, such that $r+s = k \geq 2$ is given by $r+s-1$. Hence the expression (8) can be re-written as:

$$\sum_{k\geq 1} k\, f_{k,1}(x) \;=\; \sum_{k\geq 1} \frac{k\, x^{3(k+1)}}{(1-x^{k+1})^2}. \tag{9}$$

Using the same argument as in Section 3, we can state that such a generating function is not D-finite. Developing the first terms of the generating function $h^{\alpha\wedge\beta}(x)$:

$x^6 + 2\,x^8 + 2\,x^9 + 3\,x^{10} + 11\,x^{12} + 5\,x^{14} + 10\,x^{15} + 12\,x^{16} + 20\,x^{18} + 25\,x^{20} + 16\,x^{21} + 9\,x^{22} + 51\,x^{24} + 12\,x^{25} + 11\,x^{26} + 22\,x^{27} + 39\,x^{28} + 69\,x^{30} + 46\,x^{32} + \ldots$

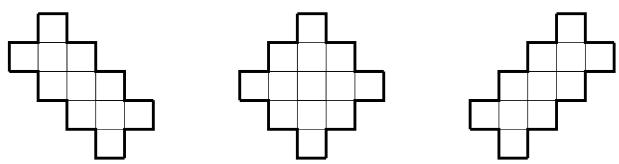

Figure 15: The 3 polyominoes in $\mathcal{H}^{\alpha\wedge\beta}$ having semi-perimeter equal to 10.

According to the statement of Proposition 4.2, a polyomino in $\mathcal{H}^{\alpha\wedge\beta}$ has semi-perimeter equal to $k(r+s)+s+k'(r+s)+r = (k+k')(r+s+1)$, with $r, s, k, k' \geq 1$. As a consequence we have that, for $n \geq 6$,

$$[x^n]\, h^{\alpha\wedge\beta}(x) \;=\; 0 \;\text{ if and only if }\; n \text{ is a prime number.}$$

Finally, we have that the generating function of $\mathcal{H}$ according to the semi-perimeter is given by

$$\frac{2\,x^6}{(1-x)^2(1-2x)^2} - \sum_{k\geq 1} \frac{k\, x^{3(k+1)}}{(1-x^{k+1})^2},$$

giving the sequence $1, 12, 44, 142, 399, 1044, 2571, 6168, 14357, 32786, \ldots$ (not in [21]).

## 5 Conclusion and further work

In this article, we studied the enumeration of two classes of pseudo-square polyominoes.

The first class we have considered consists of parallelogram polyominoes. The unicity of the decomposition of a parallelogram polyomino on pseudo-square (Prop. 3.1) leads to an interesting structural property, and then to the enumeration of the pseudo-square parallelogram polyominoes with flat bottom. The generating function (3) of this class is obtained as an infinite summation of rational functions for which we were not able to determine a closed form. We considered then the problem of enumerating *psp*-polyominoes with fixed bottom $Y$; by representing polyominoes as words of a regular language $\mathcal{L}_Y$, we gave an explicit construction of the automaton recognizing $\mathcal{L}_Y$, obtaining easily its generating function.

Our approach is a first step for understanding the general enumeration problem. However, this approach is not successful in determining a closed form of the generating function, neither in proving the (rather predictable) fact that this generating function is not *differentiably finite* (briefly, *D-finite*).

The second class we have treated consists of the pseudo-square convex polyominoes which are not parallelogram ones. We observe that there are two kinds of such polyominoes: those having one only decomposition, for which the enumeration is easy and gives a rational generating function, and those having two distinct decompositions, for which the enumeration, as in the case of *psp*-polyominoes, leads to an infinite summation of rational generating functions.

Many questions remain open concerning the enumeration of pseudo-square polyominoes, and furthermore concerning the enumeration of pseudo-hexagon polyominoes.

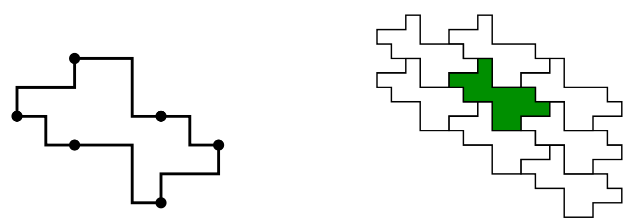

Figure 16: A pseudo hexagon and a corresponding tiling.

Another interesting problem related to the previous ones is to determine the number of the lattice periodic tilings which can be obtained by translation of one polyomino. We remark that the enumeration of exact polyominoes (i.e. polyominoes that tile the plane by translation) is closely related to the enumeration of lattice periodic tilings. Indeed an exact polyomino determines at least one (but possibly more) lattice periodic tilings: for example, the $L-$ *shaped* triomino (which is a pseudo-hexagon polyomino) generates only one lattice periodic tiling, the domino (which has two decompositions, one in pseudo-square and one in pseudo-hexagon) generates two lattice periodic tilings and the rectangle

$m \times n$ generates one exact tiling by pseudo-squares and $m+n-2$ exact tilings with pseudo-hexagons (see Figure 17).

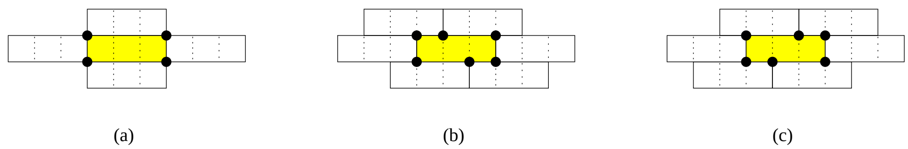


Figure 17: Periodic tilings associated to the decompositions of a triomino in a pseudo-square (a), and in two pseudo-hexagons (b) and (c).

In fact, a one-to-one correspondence can be established between the number of decompositions (in pseudo-square and pseudo-hexagons) of a given polyomino and the number of lattice tilings by this polyomino.

For instance, each *psp*-polyomino gives exactly one lattice tiling, whereas, for any $n \geq 0$ the number of different lattice tilings given by the polyominoes of $\mathcal{H}_n$ is equal to

$$|\mathcal{H}_n^\alpha| + \left|\mathcal{H}_n^\beta\right| = 2\,|\mathcal{H}_n^\alpha|\,.$$

Thus the next goal will be to find a closed formula for the number of lattice tilings by exact polyominoes, i.e.

$$\sum_{P \in \mathcal{E}} \mu(P),$$

where $\mathcal{E}$ is the set of exact polyominoes of given size, and $\mu(P)$ is the number of possible decompositions (in pseudo-squares and in pseudo-hexagons) of $P$.


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