Asymptotic Behavior in a Heap Model with Two Pieces

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Abstract

In a heap model, solid blocks, or pieces, pile up according to the Tetris game mechanism. An optimal schedule is an infinite sequence of pieces minimizing the asymptotic growth rate of the heap. In a heap model with two pieces, we prove that there always exists an optimal schedule which is balanced, either periodic or Sturmian. We also consider the model where the successive pieces are chosen at random, independently and with some given probabilities. We study the expected growth rate of the heap. For a model with two pieces, the rate is either computed explicitly or given as an infinite series. We show an application for a system of two processes sharing a resource, and we prove that a greedy schedule is not always optimal.

Key words: Optimal scheduling, timed Petri net, heap of pieces, Tetris game, (max,+) semiring, automaton with multiplicities, Sturmian word.

1 Introduction

Heap models have recently been studied as a pertinent model of discrete event systems, see Gaubert & Mairesse [19,20] and Brilman & Vincent [12,13]. They provide a good compromise between modeling power and tractability. As far as modeling is concerned, heap models are naturally associated with trace monoids, see [31]. It was proved in [20] that the behavior of a timed one-bounded Petri net can be represented using a heap model (an example appears in Figure 1). We can also mention the use of heap models in the physics of

[1] This work was partially supported by the European Community Framework IV programme through the research network ALAPEDES ("The ALgebraic Approach to Performance Evaluation of Discrete Event Systems")

surface growth, see [5]. The tractability follows essentially from the existence of a representation of the dynamic of a heap model by a (max,+) automaton, see [13,19].

A heap model is formed by a finite set of slots $\mathcal{R}$ and a finite set of pieces $\mathcal{A}$. A piece is a solid block occupying a subset of the slots and having a polyomino shape. Given a ground whose shape is determined by a vector of $\mathbb{R}^{\mathcal{R}}$ and a word $w = a_1 \cdots a_n \in \mathcal{A}^*$, we consider the heap obtained by piling up the pieces $a_1, \ldots, a_n$ in this order, starting from the ground, and according to the Tetris game mechanism. That is, pieces are subject to vertical translations and occupy the lowest possible position above the ground and previously piled up pieces. Let $y(w)$ be the height of the heap w. We define the *optimal growth rate* as $\rho_{\min} = \liminf_n \min_{w \in \mathcal{A}^n} y(w)/n$. An *optimal schedule* is an infinite word $u \in \mathcal{A}^\omega$ such that $\lim_n y(u[n])/n = \rho_{\min}$, where $u[n]$ is the prefix of length n of u. An optimal schedule exists under minimal conditions (Proposition 4). We can define similarly the quantity $\rho_{\max}$ and the notion of *worst schedule*. The problem of finding a worst schedule is completely solved, see [17,19]. Finding an optimal schedule is more difficult, the reason being the non-compatibility of the minimization with the $(\max, +)$ dynamic of the model. In [21], it is proved that if the heights of the pieces are rational, then there exists a periodic optimal schedule. If we remove the rationality assumption, the problem becomes more complicated. Here we prove, and this is the main result of the paper, that in a heap model with two pieces, there always exists an optimal schedule which is balanced, either periodic or Sturmian. We characterize the cases where the optimal is periodic and the ones where it is Sturmian. The proof is constructive, providing an explicit optimal schedule.

As will be detailed below, a heap model can be represented using a specific type of (max,+) automaton, called a *heap automaton*. A natural question is the following: Given a general (max,+) automaton over a two letter alphabet, does there always exist an optimal schedule which is balanced (for an automaton defined by the triple (α, μ, β), set $y(w) = \alpha\mu(w)\beta$ and define an optimal schedule as above)? The answer to this question is no, which emphasizes the specificity of heap automata among (max,+) automata. A counter-example is provided in Figure 4.

We also consider random words obtained by choosing successive pieces independently, with some given distribution. We denote by ρ_E the average growth rate of the heap. Computing ρ_E is in general even more difficult than computing $\rho_{\min}$. In [21], ρ_E is explicitly computed if the heights of the pieces are rational and if no two pieces occupy disjoint sets of slots. Here, for models with two pieces, we obtain an explicit formula for ρ_E in all cases but one where ρ_E is given as an infinite series.

To further motivate this work, we present a manufacturing model studied by

Gaujal & al [23,22]. There are two types of tasks to be performed on the same

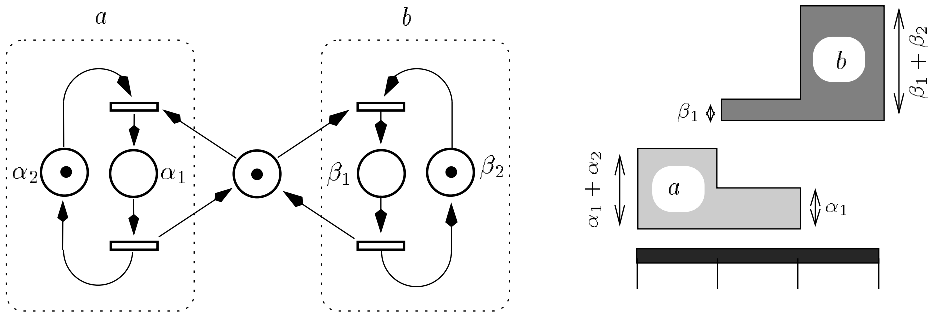

Fig. 1. One-bounded Petri net and the associated heap model.

machine used in mutual exclusion. Each task is cyclic and a cycle is constituted by two successive activities: one that requires the machine (durations: α_1 and β_1 respectively) and one that does not (durations: α_2 and β_2 respectively). Think for instance of the two activities as being the processing and the packing. This jobshop can be represented by the timed one-bounded Petri net of Figure 1. The durations $\alpha_1, \alpha_2, \beta_1$ and β_2 are the holding times of the places. As detailed in [20], an equivalent description is possible using the heap model represented in Figure 1. The height of a heap $a_1 \cdots a_n, a_i \in \{a, b\}$, corresponds to the total execution time of the sequence of tasks $a_1, \ldots, a_n$ executed in this order. An infinite schedule is optimal if it minimizes the average height of the heap, or equivalently if it maximizes the throughput of the Petri net. We do not make any restriction on the schedules we consider. In particular we do not impose a frequency for tasks a and b. As a justification, imagine for instance that the two tasks correspond to two different ways of processing the same object. We prove in §7.4 that if $\alpha_1 = \beta_1 = 0, \alpha_2 > 0, \beta_2 > 0, \alpha_2/\beta_2 \notin \mathbb{Q}$, then there is a Sturmian optimal schedule; otherwise there exists a balanced periodic optimal schedule. We also show in §7.5 that the greedy schedule is not always optimal.

Assume now that in the model of Figure 1, the successive tasks to be executed are chosen at random, independently, and with some probabilities $p(a)$ and $p(b)$. If α_1 or β_1 is strictly positive, then we obtain an exact formula for ρ_E. It enables in particular to maximize the throughput over all possible choices for $p(a)$ and $p(b)$, see §8 for an example.

Let us compare the results of this paper with other cases where optimality is attained via balance. In Hajek [25], there is a flow of arriving customers to be dispatched between two queues and the problem is to find the optimal behavior under a ratio constraint for the routings. The author introduces the notion of multimodularity, a discrete version of convexity, and proves that a multimodular objective function is minimized by balanced schedules. Variants and extensions to other open queueing or Petri net models have been carried out in [1,2], still using multimodularity. In a heap model however, one can prove that the heights are not multimodular. In [22,23], the authors consider the model of Figure 1. They study the optimal behavior and the optimal behavior under a frequency constraint for the letters. Balanced schedules are shown to be optimal and the proofs are based on various properties of these sequences. We consider a more general model. For the unconstrained problem, we prove in Theorem 14 that balanced schedules are again optimal. On the other hand, under frequency constraints, we show in §7.6 that optimality is not attained via balanced words anymore. Our methods of proof are completely different from the ones mentioned above.

The paper is organized as follows. In §2 and §3, we define precisely the model and the problems considered. We prove the existence of optimal schedules under some mild conditions in §3.1. In §4, we recall some properties of balanced words. We introduce in §5 the notions of completion of contours and completion of pieces in a heap model. We prove in §6 that it is always possible to study a heap model with two pieces by considering an associated model with at most 3 slots. We provide an enumeration of all the possible simplified models: there are 4 cases. In §7.1-7.4, we prove the result on optimal schedules, recalled above, by considering the four cases one by one. Greedy scheduling is discussed in §7.5, and ratio constraints in §7.6. In §8, we study the average growth rate.

2 Heap Model

Consider a finite set $\mathcal{R}$ of *slots* and a finite set $\mathcal{A}$ of *pieces*. A piece $a \in \mathcal{A}$ is a rigid (possibly non-connected) "block" occupying a subset $R(a)$ of the slots. It has a lower contour and an upper contour which are represented by two row vectors $l(a)$ and $u(a)$ in $(\mathbb{R}\cup\{-\infty\})^{\mathcal{R}}$ with the convention $l(a)_r = u(a)_r = -\infty$ if $r \notin R(a)$. They satisfy $u(a) \geq l(a)$. We assume that each piece occupies at least one slot, $\forall a \in \mathcal{A}, R(a) \neq \emptyset$, and that each slot is occupied by at least one piece, $\forall r \in \mathcal{R}, \exists a \in \mathcal{A}, r \in R(a)$. The shape of the ground is given by a vector $I \in \mathbb{R}^{\mathcal{R}}$. The 6-tuple $\mathcal{H} = (\mathcal{A}, \mathcal{R}, R, u, l, I)$ constitutes a *heap model.*

The mechanism of the building of heaps was described in the introduction. It is best understood visually and on an example.

Example 1 *We consider the following heap model.*

- $\mathcal{A} = \{a, b\}$, $\mathcal{R} = \{1, 2, 3\}$, $I = (0, 0, 0)$;
- $R(a) = \{1, 2\}$, $R(b) = \{2, 3\}$;
- $u(a) = (\alpha_1 + \alpha_2, \alpha_1, -\infty)$, $l(a) = (0, 0, -\infty)$,
- $u(b) = (-\infty, \beta_1, \beta_1 + \beta_2)$, $l(b) = (-\infty, 0, 0)$,

where $\alpha_1, \alpha_2, \beta_1$ *and* β_2 *are strictly positive reals. We have represented, in*

Fig. 2, the heap associated with the word $w = ababa$.

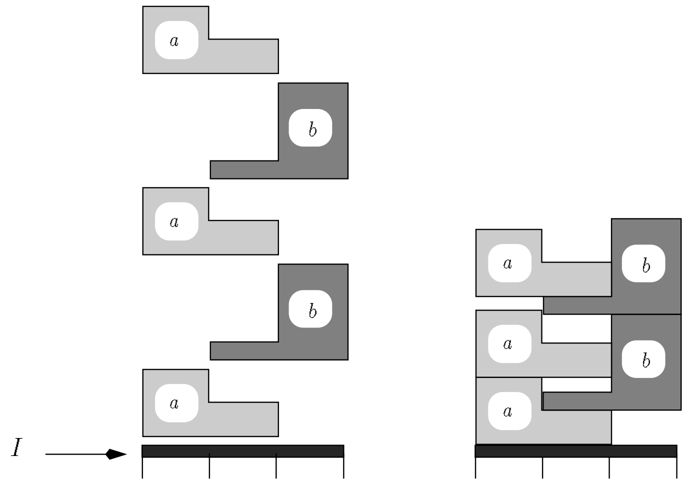

Fig. 2. Heap associated with the word *ababa*.

We recall some standard definitions and notations. We denote by $\mathbf{1}\{A\}$ the function which takes value 1 if A is true and 0 if A is false. We denote by $\mathbb{R}_+$ the set of non-negative reals, and by $\mathbb{N}^*$ and $\mathbb{R}^*$ the sets $\mathbb{N}\backslash\{0\}$ and $\mathbb{R}\backslash\{0\}$. Let $\mathcal{A}$ be a finite set (*alphabet*). We denote by $\mathcal{A}^*$ the *free monoid* on $\mathcal{A}$, that is, the set of *(finite) words* equipped with concatenation. The *empty word* is denoted by e. The length of a word w is denoted by $|w|$ and we write $|w|_a$ for the number of occurrences of the letter a in w. We denote by $alph(w)$ the set of distinct letters appearing in w. An *infinite word* (or *sequence*) is a mapping $u : \mathbb{N}^* \to \mathcal{A}$. The set of infinite words is denoted by $\mathcal{A}^\omega$. An infinite word $u = u_1 u_2 \cdots$ is *periodic* if there exists $l \in \mathbb{N}^*$ such that $u_{i+l} = u_i, \forall i \in \mathbb{N}^*$. In this case, we write $u = (u_1 \cdots u_l)^\omega$. We denote by $u[n] = u_1 u_2 \cdots u_n$ the prefix of length n of u.

When $\mathcal{A}$ is the set of pieces of a heap model, (infinite) words will also be called (infinite) *schedules*. We also interpret a word $w \in \mathcal{A}^*$ as a heap, i.e. as a sequence of pieces piled up in the order given by the word.

The *upper contour* of the heap w is a row vector $x_{\mathcal{H}}(w)$ in $\mathbb{R}^{\mathcal{R}}$, where $x_{\mathcal{H}}(w)_r$ is the height of the heap on slot r. By convention, $x_{\mathcal{H}}(e) = I$, the shape of the ground. The *height* of the heap w is

$$y_{\mathcal{H}}(w) = \max_{r \in \mathcal{R}} x_{\mathcal{H}}(w)_r \ . \tag{1}$$

We recall that a set K equipped with two operations $\oplus$ and $\otimes$ is a *semiring* if $\oplus$ is associative and commutative, $\otimes$ is associative and distributive with respect to $\oplus$, there is a zero element $\mathbb{0}$ ($a \oplus \mathbb{0} = a, a \otimes \mathbb{0} = \mathbb{0} \otimes a = \mathbb{0}$) and a unit element $\mathbb{1}$ ($a \otimes \mathbb{1} = \mathbb{1} \otimes a = a$).

The set $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \max, +)$ is a semiring, called the *(max,+) semiring*. From now on, we use the semiring notations: $\oplus = \max, \otimes = +, \mathbb{0} = -\infty$ and $\mathbb{1} = 0$. The semiring $\mathbb{R}_{\min}$ is obtained from $\mathbb{R}_{\max}$ by replacing max by min and $-\infty$ by $+\infty$. The subsemiring $\mathbb{B} = (\mathbb{0}, \mathbb{1}, \oplus, \otimes)$ is the *Boolean* semiring.

We use the matrix and vector operations induced by the semiring structure. For matrices A, B of appropriate sizes, $(A \oplus B)_{ij} = A_{ij} \oplus B_{ij} = \max(A_{ij}, B_{ij})$, $(A \otimes B)_{ij} = \bigoplus_k A_{ik} \otimes B_{kj} = \max_k(A_{ik} + B_{kj})$, and for a scalar a, $(a \otimes A)_{ij} = a \otimes A_{ij} = a + A_{ij}$. We usually omit the $\otimes$ sign, writing for instance AB instead of $A \otimes B$. On the other hand, the operations denoted by $+, -, \times$ and $/$ always have to be interpreted in the conventional algebra. We define the 'pseudo-norm' $|A|_\oplus = \max_{ij} A_{ij}$. We denote by $\mathbb{0}$, resp. $\mathbb{1}$, the vector or matrix whose elements are all equal to $\mathbb{0}$, resp. $\mathbb{1}$ (with the dimension depending on the context).

For matrices A and B of appropriate sizes, the proof of the following inequality is immediate:

$$|AB|_\oplus \leq |A|_\oplus \otimes |B|_\oplus \,. \tag{2}$$

For matrices U, V and A of appropriate sizes and such that all the entries of U, V, UA and VA are different from $\mathbb{0}$, the following non-expansiveness inequality holds:

$$|UA - VA|_\oplus \leq |U - V|_\oplus \,. \tag{3}$$

Given an alphabet $\mathcal{A}$, a *(max,+) automaton* of dimension k is a triple (α, μ, β), where $\alpha \in \mathbb{R}_{\max}^{1\times k}$ and $\beta \in \mathbb{R}_{\max}^{k\times 1}$, are the initial and final vectors and where $\mu : \mathcal{A}^* \to \mathbb{R}_{\max}^{k\times k}$ is a monoid morphism. The morphism μ is entirely defined by the matrices $\mu(a), a \in \mathcal{A}$, and for $w = w_1 \cdots w_n$, we have $\mu(w) = \mu(w_1) \cdots \mu(w_n)$ (product of matrices in $\mathbb{R}_{\max}$). The map $y : \mathcal{A}^* \to \mathbb{R}_{\max}$, $y(w) = \alpha\mu(w)\beta$ is said to be *recognized* by the (max,+) automaton. A (max,+) automaton is a specialization to $\mathbb{R}_{\max}$ of the classical notion of an automaton with multiplicities, see [8,16].

An automaton (α, μ, β) of dimension k over the alphabet $\mathcal{A}$ is represented graphically by a labelled digraph. The graph has k nodes; if $\mu(a)_{ij} > \mathbb{0}$ then there is an arc between nodes i and j with labels a and $\mu(a)_{ij}$; if $\alpha_i > \mathbb{0}$ then there is an ingoing arrow at node i with label α_i and if $\beta_j > \mathbb{0}$ then there is an outgoing arrow at node j with label β_j. Examples appear in Figures 9,10 or 11.

For each piece a of a heap model $\mathcal{H}$, we define the matrix $\mathcal{M}(a) \in \mathbb{R}_{\max}^{\mathcal{R}}$ by

$$\mathcal{M}(a)_{sr} = \begin{cases} \mathbb{1} & \text{if } s = r, r \notin R(a), \\ u(a)_r - l(a)_s & \text{if } r \in R(a), s \in R(a), \\ \mathbb{0} & \text{otherwise.} \end{cases} \tag{4}$$

Example 2 *In the model considered in Figure 1 and Example 1, the matrices associated with the pieces are*

$$\mathcal{M}(a) = \begin{pmatrix} \alpha_1\alpha_2 & \alpha_1 & \mathbb{0} \\ \alpha_1\alpha_2 & \alpha_1 & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathbb{1} \end{pmatrix}, \quad \mathcal{M}(b) = \begin{pmatrix} \mathbb{1} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \beta_1 & \beta_1\beta_2 \\ \mathbb{0} & \beta_1 & \beta_1\beta_2 \end{pmatrix}.$$

The entries have to be interpreted in $\mathbb{R}_{\max}$.

Variants of Theorem 3 are proved in [13,19,20].

Theorem 3 *Let* $\mathcal{H} = (\mathcal{A}, \mathcal{R}, R, u, l, I)$ *be a heap model. For a word* $w = w_1 \cdots w_n$, *the upper contour and the height of the heap satisfy (products in* $\mathbb{R}_{\max}$*)*

$$\begin{aligned} x_{\mathcal{H}}(w) &= I\mathcal{M}(w_1)\cdots\mathcal{M}(w_n)\,, \\ y_{\mathcal{H}}(w) &= I\mathcal{M}(w_1)\cdots\mathcal{M}(w_n)\mathbb{1}\,. \end{aligned} \tag{5}$$

More formally, $y_{\mathcal{H}}$ *is recognized by the (max,+) automaton* $(I, \mathcal{M}, \mathbb{1})$.

From now on, we identify the heap model and the associated (max,+) automaton, writing either $\mathcal{H} = (\mathcal{A}, \mathcal{R}, R, u, l, I)$ or $\mathcal{H} = (I, \mathcal{M}, \mathbb{1})$. We also call $\mathcal{H}$ a *heap automaton.*

3 Asymptotic Behavior

Consider a (max,+) automaton $\mathcal{U} = (\alpha, \mu, \beta)$ and its recognized map y. We define the *optimal growth rate* (in $\mathbb{R} \cup \{-\infty\}$) as:

$$\rho_{\min}(\mathcal{U}) = \liminf_{n\to+\infty} \frac{1}{n} \min_{w\in\mathcal{A}^n} y(w)\,. \tag{6}$$

An *optimal schedule* is a word $w \in \mathcal{A}^\omega$ such that $\lim_n y(w[n])/n = \rho_{\min}(\mathcal{U})$.

We define the *worst growth rate* as $\rho_{\max}(\mathcal{U}) = \limsup_{n\to+\infty} \max_{w\in\mathcal{A}^n} y(w)/n$. A *worst schedule* is defined accordingly.

Consider a probability law $\{p(a), a \in \mathcal{A}\}$ ($p(a) \in [0,1]$, $\sum_{a\in\mathcal{A}} p(a) = 1$). Random words are built by choosing the successive letters independently and according to this law. Let $p(w), |w| = n$, be the probability for a random word of length n to be w. We have $p(w) = p(w_1) \times p(w_2) \times \cdots \times p(w_n)$ if $w = w_1 w_2 \cdots w_n$. When it exists, we define the *average growth rate* as:

$$\rho_E(\mathcal{U}) = \lim_{n \to +\infty} \frac{1}{n} \sum_{w \in \mathcal{A}^n} p(w) \times y(w) . \tag{7}$$

The *optimal problem* consists in evaluating $\rho_{\min}(\mathcal{U})$ and finding an optimal schedule. The *worst case problem* consists in evaluating $\rho_{\max}(\mathcal{U})$ and finding a worst schedule. The *average case problem* consists in evaluating $\rho_E(\mathcal{U})$.

When we consider a heap automaton $\mathcal{H}$, the limits $\rho_{\min}(\mathcal{H}), \rho_{\max}(\mathcal{H})$ and $\rho_E(\mathcal{H})$ correspond respectively to the minimal, maximal and average asymptotic growth rate of a heap.

3.1 Preliminary results

We consider the optimal problem first. It follows from (2) that $\min_{|w|=n+m} |\mu(w)|_\oplus \leq \min_{|w|=n} |\mu(w)|_\oplus + \min_{|w|=m} |\mu(w)|_\oplus$. As a consequence of the subadditive theorem, we have

$$\lim_n \frac{1}{n} \min_{|w|=n} |\mu(w)|_\oplus = \inf_n \frac{1}{n} \min_{|w|=n} |\mu(w)|_\oplus = \rho . \tag{8}$$

We also have for all $w \in \mathcal{A}^*$,

$$|\mu(w)|_\oplus \otimes \min_i \alpha_i \otimes \min_i \beta_i \;\leq\; \alpha\mu(w)\beta \;\leq\; |\mu(w)|_\oplus \otimes |\alpha|_\oplus \otimes |\beta|_\oplus . \tag{9}$$

When $\alpha_i > \mathbb{0}, \beta_i > \mathbb{0}, \forall i$, we deduce that $\rho_{\min}(\mathcal{U}) = \rho$ and that the lim inf is a limit in (6).

Proposition 4 *Let $\mathcal{U} = (\alpha, \mu, \beta)$ be a (max,+) automaton such that $\forall i, \alpha_i > \mathbb{0}$, $\beta_i > \mathbb{0}$, and such that $\rho_{\min}(\mathcal{U}) \neq \mathbb{0}$. Then there exists an optimal schedule.*

PROOF. It follows from (9) that the automata (α, μ, β) and $(\mathbb{1}, \mu, \mathbb{1})$ have the same optimal schedules (if any). Since $\rho_{\min} \neq \mathbb{0}$, we deduce from (8) that for all $k \in \mathbb{N}^*$, there exists $w(k) \in \mathcal{A}^* \setminus \{e\}$ such that

$$|w(k)| \times \rho_{\min} \leq |\mu(w(k))|_\oplus \leq |w(k)| \times (\rho_{\min} + \frac{1}{k}) .$$

By the subadditive inequality (2), we then have, for all $l \in \mathbb{N}^*$,

$$|w(k)^l| \times \rho_{\min} \leq |\mu(w(k)^l)|_\oplus \leq |w(k)^l| \times (\rho_{\min} + \frac{1}{k}) . \tag{10}$$

Now define $\tilde{w}(k) = w(k)^{k|w(k+1)|}$ and consider the infinite word $\tilde{w} = \tilde{w}(1)\tilde{w}(2)\cdots\tilde{w}(k)\cdots$ obtained by concatenation of the words $\tilde{w}(k)$. We consider the prefix of length n of $\tilde{w}$ for an arbitrary $n \in \mathbb{N}^*$. There exists $k_n \in \mathbb{N}^*$ such that

$$\tilde{w}[n] = \tilde{w}(1)\cdots\tilde{w}(k_n)w(k_n+1)^l u \ ,$$

where $0 \leq l < (k_n+1)|w(k_n+2)|$ and where u is a prefix of $w(k_n+1)$. Using (2) and (10), we get

$$\begin{aligned}\rho_{\min} \ \leq \ \frac{|\mu(\tilde{w}[n])|_\oplus}{n} &\leq \sum_{i=1}^{k_n} \frac{|\mu(\tilde{w}(i))|_\oplus}{n} + \frac{|\mu(w(k_n+1)^l)|_\oplus}{n} + \frac{|\mu(u)|_\oplus}{n} \\ &\leq \rho_{\min} + \sum_{i=1}^{k_n} \frac{|\tilde{w}(i)|}{ni} + \frac{|w(k_n+1)^l|}{n(k_n+1)} + \frac{|\mu(u)|_\oplus}{n} \ . \qquad (11)\end{aligned}$$

Obviously, k_n is an increasing function of n and $\lim_{n\to+\infty} k_n = +\infty$. Hence, we obtain that:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}^*, \forall n \geq N, \ \ \sum_{i=1}^{k_n} \frac{|\tilde{w}(i)|}{ni} + \frac{|w(k_n+1)^l|}{n(k_n+1)} \leq \varepsilon \ . \qquad (12)$$

Let us take care of the last term on the right-hand side of (11). Note that $|u| \leq |w(k_n+1)|$ and $n = |\tilde{w}[n]| \geq |\tilde{w}(k_n)| = k_n|w(k_n+1)|$. It implies that

$$\frac{|\mu(u)|_\oplus}{n} \leq \frac{|u| \bigoplus_{a\in\mathcal{A}} |\mu(a)|_\oplus}{n} \leq \frac{\bigoplus_{a\in\mathcal{A}} |\mu(a)|_\oplus}{k_n} \ . \qquad (13)$$

Starting from (11) and using (12) and (13), we obtain that

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}^*, \forall n \geq N, \ \ \rho_{\min} \ \leq \ \frac{|\mu(\tilde{w}[n])|_\oplus}{n} \leq \rho_{\min} + 2\varepsilon \ .$$

It completes the proof.

We now consider the worst case problem. As above, if $\forall i, \alpha_i > \mathbb{0}, \beta_i > \mathbb{0}$, then the lim sup is a limit in the definition of $\rho_{\max}$. As opposed to the optimal case, the worst case problem is completely solved. We recall the main result; it is taken from [17] and it follows from the (max,+) spectral theorem (the most famous and often rediscovered result in the (max,+) semiring, see [15,4,28] and the references therein).

Proposition 5 *Let $\mathcal{U} = (\alpha, \mu, \beta)$ be a trim (see §3.2) (max,+) automaton of dimension k. Then, $\rho_{\max}(\mathcal{U})$ is equal to $\rho_{\max}(M)$, the maximal eigenvalue of* *the matrix* $M = \bigoplus_{a\in\mathcal{A}} \mu(a)$. *That is*

$$\rho_{\max}(\mathcal{U}) = \bigoplus_{1\le l\le k} \bigoplus_{i_1,\dots,i_l} (M_{i_1i_2}\cdots M_{i_li_1})^{1/l} = \max_{1\le l\le k} \max_{i_1,\dots,i_l} \frac{M_{i_1i_2}+\cdots+M_{i_li_1}}{l} .$$

Let a_{ij} *be such that* $\mu(a_{ij})_{ij} = \max_{a\in\mathcal{A}} \mu(a)_{ij}$ *and let* $(i_1,\dots,i_l)$ *be such that* $(M_{i_1i_2}+\cdots+M_{i_li_1})/l = \rho_{\max}(M)$ *(we say that* $(i_1,\dots,i_l)$ *is a* maximal mean weight *circuit of* M*). Then* $(a_{i_1i_2}\cdots a_{i_li_1})^\omega$ *is a worst schedule.*

In the case of a heap automaton, there exists a worst schedule of the form u^ω, where the period u is such that $\forall a\in\mathcal{A}, |u|_a \le 1$. For a heap automaton with two pieces (a and b), a worst schedule can always be found among a^ω, b^ω and $(ab)^\omega$. An example where the worst schedule is indeed $(ab)^\omega$ appears in Figure 18.

3.2 Deterministic automaton

A (max,+) automaton (α,μ,β) is *trim* if for each state i, there exist words u and v such that $\alpha\mu(u)_i > \mathbb{0}$ and $\mu(v)\beta_i > \mathbb{0}$. It is *deterministic* if there exists exactly one i such that $\alpha_i > \mathbb{0}$; and if for all letter a and for all i, there exists at most one j such that $\mu(a)_{ij} > \mathbb{0}$. It is *complete* if for all letter a and for all i, there exists at least one j such that $\mu(a)_{ij} > \mathbb{0}$.

A heap automaton is deterministic if and only if there is a single slot. On the other hand, a heap automaton is obviously always trim and complete. In the course of the paper, we consider other types of (max,+) automata: Cayley and contour-completed automata. These automata will be deterministic, trim and complete.

Let $\mathcal{U} = (\alpha,\mu,\beta)$ be a deterministic and trim (max,+) automaton over the alphabet $\mathcal{A}$. Let $\mathcal{U}'$ be the (min,+) automaton defined by the same triple (with $\mathbb{0} = +\infty$). Let $y_{\mathcal{U}}$ and $y_{\mathcal{U}'}$ be the maps recognized by $\mathcal{U}$ and $\mathcal{U}'$ respectively. Since $\mathcal{U}$ is deterministic, it follows that $y_{\mathcal{U}'}(w) = y_{\mathcal{U}}(w)$ if $y_{\mathcal{U}}(w) \neq -\infty$ and $y_{\mathcal{U}'}(w) = +\infty$ if $y_{\mathcal{U}}(w) = -\infty$. Defining the (min,+) matrix $N = \min_{a\in\mathcal{A}} \mu(a)$ and applying the (min,+) version of Proposition 5 (replace max by min everywhere in the statement of the Proposition), we get that

$$\rho_{\min}(\mathcal{U}) = \rho_{\min}(\mathcal{U}') = \rho_{\min}(N) , \tag{14}$$

the minimal eigenvalue of N. Also if $(i_1,\dots,i_l)$ is a minimal mean weight circuit, then $(a_{i_1i_2}\cdots a_{i_li_1})^\omega$ is an optimal schedule.

Proposition 6 *Let* $\mathcal{U} = (\alpha,\mu,\beta)$ *be a deterministic, complete and trim (max,+) automaton over the alphabet* $\mathcal{A}$*. Assume that* $M = \bigoplus_{a\in\mathcal{A}} \mu(a)$ *is* *an irreducible matrix (i.e. $\forall i, j, \exists k, M_{ij}^k > \mathbb{0}$). We define the $(\mathbb{R}_+, +, \times)$ matrix P by $P_{ij} = \sum_{a\in\mathcal{A}} p(a) \times \mathbf{1}\{\mu(a)_{ij} > \mathbb{0}\}$. Let π be the unique vector satisfying $\pi \times P = \pi$ and $\sum_i \pi(i) = 1$. The expected growth rate is $\rho_E(\mathcal{U}) = \sum_i \pi(i) \left(\sum_{j,a} p(a)\mu(a)_{ij}\mathbf{1}\{\mu(a)_{ij} > \mathbb{0}\}\right)$ (the products are the usual ones).*

Proposition 6 is proved in [17]. It follows from standard results in Markov chain theory (P is the transition matrix and π is the stationary distribution). A consequence of Proposition 6 is that $\rho_E(\mathcal{U})$ can be written formally as a rational fraction of the probabilities of the letters. That is $\rho_E(\mathcal{U}) = R/S$ and R and S are real polynomials over the commuting indeterminates $p(a), a \in \mathcal{A}$. More generally, it is possible, under the assumptions of Prop. 6, to obtain the formal power series $s = \sum_{n\in\mathbb{N}}(\sum_{w\in\mathcal{A}^n} p(w) \times y(w))x^n$ as a rational fraction (over the indeterminates $x, p(a), a \in \mathcal{A}$), see for instance [8].

Finitely distant automata. Two (max,+) automata $\mathcal{U} = (\alpha, \mu, \beta)$ and $\mathcal{V} = (\gamma, \nu, \delta)$ defined over the same alphabet $\mathcal{A}$ are said to be *finitely distant* if

$$\begin{cases} \alpha\mu(w)\beta = \mathbb{0} \Longleftrightarrow \gamma\nu(w)\delta = \mathbb{0}\,; \\ \exists M < \infty, \ \ \sup_{w,\alpha\mu(w)\beta\neq\mathbb{0}} |\alpha\mu(w)\beta - \gamma\nu(w)\delta| \leq M\,. \end{cases} \tag{15}$$

Two heap automata $(I, \mathcal{M}, \mathbb{1})$ and $(I', \mathcal{M}, \mathbb{1})$ are finitely distant. Indeed, according to (3), we have

$$I\mathcal{M}(w)\mathbb{1} - I'\mathcal{M}(w)\mathbb{1} \leq |I\mathcal{M}(w) - I'\mathcal{M}(w)|_\oplus \leq |I - I'|_\oplus\,.$$

The asymptotic problems are equivalent for two finitely distant automata $\mathcal{U}$ and $\mathcal{V}$. That is $\rho_E(\mathcal{U}) = \rho_E(\mathcal{V})$, $\rho_{\min}(\mathcal{U}) = \rho_{\min}(\mathcal{V})$ and optimal schedules coincide.

Since most heap automata are not deterministic, we can not apply the results in (14) and Proposition 6 directly to them. We often use the following procedure: Given a (max,+) automaton, find a deterministic, trim, and finitely distant automaton, then apply the above results to the new automaton.

4 Balanced Words

Balanced and Sturmian words appear under various names and in various areas like number theory and continued fractions [29], physics and quasi-crystals [24] or discrete event systems [25,22]. For reference papers on the subject, see [7,9].

A finite word u is a *factor* of a (finite or infinite) word $w = w_1w_2\cdots$ if u

is a finite subsequence of consecutive letters in w, i.e. $u = w_i w_{i+1} \cdots w_{i+n-1}$ for some i and n. A (finite or infinite) word w is *balanced* if $|\,|u|_a - |v|_a| \leq 1$ for all letter a and for all factors u, v of w such that $|u| = |v|$. The balanced words are the ones in which the letters are the most regularly distributed. The shortest non-balanced word is $aabb$.

An infinite word u is ultimately periodic if there exist $n \in \mathbb{N}^*$ and $l \in \mathbb{N}^*$ such that $u_{i+l} = u_i$ for all $i \geq n$. A *Sturmian word* is an infinite word over a two letters alphabet which is balanced and not ultimately periodic.

We now define jump words. Let us consider $\alpha_1, \alpha_2 \in \mathbb{R}_+^*$ and $\gamma \in \mathbb{R}_+, \gamma < \alpha_2$. We label the points $\{n\alpha_1, n \in \mathbb{N}^*\}$ by a, and the points $\{n\alpha_2 + \gamma, n \in \mathbb{N}^*\}$ by b. Let us consider the set $\{n\alpha_1, n \in \mathbb{N}^*\} \cup \{n\alpha_2 + \gamma, n \in \mathbb{N}^*\}$ in its natural order and the corresponding sequence of labels. Each time there is a double point, we choose to read a before b. We obtain the *jump word* with characteristics $(\alpha_1, \alpha_2, \gamma)$. Jump words are balanced. If α_1/α_2 is rational then w is periodic; if α_1/α_2 is irrational then w is Sturmian.

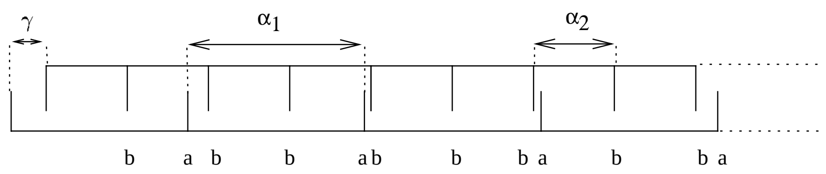

Fig. 3. Representation of the jump word $(\alpha_1, \alpha_2, \gamma)$.

It is also possible to define words as above except that we read b before a whenever there is a double point. These words are still balanced and we still call them *jump words* (below, when necessary, we will precise what is the convention used for double points).

A more common but similar description of jump words uses *cutting sequences*. There exists an explicit arithmetic formula to compute the n-th letter in a given jump word (using the so-called *mechanical* characterization, see [9]).

Optimal schedules and balanced words. We prove in Theorem 14 that in a heap model with two pieces, there always exist an optimal schedule which is balanced. If we still consider a two letter alphabet but a general (max,+) automaton, then this is not true anymore. The counter-example below was suggested to us by Thierry Bousch [10]. Consider the deterministic (max,+) automaton $(\delta, \mu, \mathbb{1})$ represented in Figure 4. It is easy to check that $\rho_{\min} = \mathbb{1}$ and that an optimal schedule is the non-balanced word $(aabb)^\omega$. No balanced word is optimal in this example.

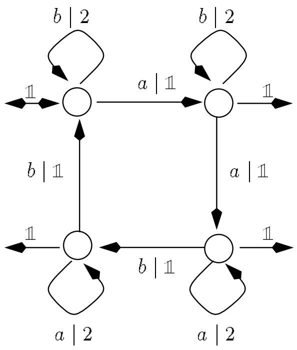

Fig. 4. (Max,+) automaton with no balanced optimal schedules.

5 Completion of Profiles and Pieces

5.1 Cayley automaton

Given A in $\mathbb{R}_{\max}^{k\times l}$, we define $\pi(A)$ in $\mathbb{R}_{\max}^{k\times l}$ by $\pi(A)_{ij} = A_{ij} - |A|_\oplus$ if $|A|_\oplus \neq \mathbb{0}$ and $\pi(A) = A = \mathbb{0}$ otherwise. We have $|\pi(A)|_\oplus = \mathbb{1}$ (except if $A = \mathbb{0}$). We say that $\pi(A)$ is the *normalized* matrix associated with A.

Let us consider a (max,+) automaton $\mathcal{U} = (\alpha, \mu, \beta)$ over the alphabet $\mathcal{A}$. We define

$$\pi(\mathcal{U}) = \{\pi(\alpha\mu(w)), w \in \mathcal{A}^*\} . \tag{16}$$

In the case of a heap automaton $\mathcal{H}$, $\pi(\mathcal{H})$ is the set of *normalized* upper contours.

Assume that $\pi(\mathcal{U})$ is finite. Then we define the *Cayley automaton* of $\mathcal{U}$ as follows. It is the deterministic (max,+) automaton (δ, ν, γ) of dimension $\pi(\mathcal{U})$ over the alphabet $\mathcal{A}$, where for $u, v \in \pi(\mathcal{U}), a \in \mathcal{A}$,

$$\delta_u = \begin{cases} |\alpha|_\oplus & \text{if } u = \pi(\alpha) \\ \mathbb{0} & \text{otherwise} \end{cases}, \quad \nu(a)_{uv} = \begin{cases} |u\mu(a)|_\oplus & \text{if } \pi(u\mu(a)) = v \\ \mathbb{0} & \text{otherwise}\,. \end{cases},$$

and $\gamma_u = u\beta$. It follows from this definition that for $w \in \mathcal{A}^*$, $\nu(w)_{uv} = |u\mu(w)|_\oplus$ if $\pi(u\mu(w)) = v$ and $\nu(w)_{uv} = \mathbb{0}$ otherwise. Hence we have

$$\begin{aligned} \delta\nu(w)\gamma &= \delta_{\pi(\alpha)}\nu(w)_{\pi(\alpha)\pi(\pi(\alpha)\mu(w))}\gamma_{\pi(\pi(\alpha)\mu(w))} \\ &= |\alpha|_\oplus\ |\pi(\alpha)\mu(w)|_\oplus\ \pi(\pi(\alpha)\mu(w))\beta \\ &= |\alpha\mu(w)|_\oplus\ \pi(\alpha\mu(w))\beta \ = \ \alpha\mu(w)\beta\,. \end{aligned} \tag{17}$$

We just proved that the automaton $\mathcal{U}$ and its Cayley automaton recognize the same map (see also [17]).

The dimension of the Cayley automaton is in general much larger than the one of $\mathcal{U}$. However, it is deterministic, complete, and assuming for instance that $\forall i, \beta_i > \mathbb{0}$, it is also trim. In particular when $\mathcal{H}$ is a heap automaton and $\pi(\mathcal{H})$ is finite, then the Cayley automaton is deterministic, complete and trim. The Cayley automaton is used in §7.2.

The procedure described above is similar to the classical determinization algorithm for Boolean automata. The difference is of course that $\pi(\mathcal{U})$ is always finite in the Boolean case.

5.2 Contour-completed automaton

Given a heap model $\mathcal{H}$, it is easy to see that $\pi(\mathcal{H})$ is infinite as soon as there exist two pieces a and b whose slots are not the same. This motivated the introduction in [21] of the refined notion of *normalized completed contours*. In some cases, the set of such contours will be finite whereas $\pi(\mathcal{H})$ is infinite. Here, we recall only the results that will be needed. For details, and in particular for an algebraic definition of completion in terms of *residuation*, see [21].

Let us consider a heap model $\mathcal{H} = (\mathcal{A}, \mathcal{R}, R, u, l, I)$, also described as the heap automaton $\mathcal{H} = (I, \mathcal{M}, \mathbb{1})$. We associate with the piece $a \in \mathcal{A}$, the *upper contour piece* $\overline{a}$ and the *lower contour piece* $\underline{a}$ defined as follows

$$l(\overline{a}) = u(a), u(\overline{a}) = u(a), \text{ and } l(\underline{a}) = l(a), u(\underline{a}) = l(a)\,.$$

We still denote by $\mathcal{M}(\overline{a}), \mathcal{M}(\underline{a})$, the matrices defined as in (4) and associated with the new pieces $\overline{a}, \underline{a}$.

An example of upper and lower contour pieces is provided in Figure 5. For clarity, pieces of height 0 are represented by a thick line.

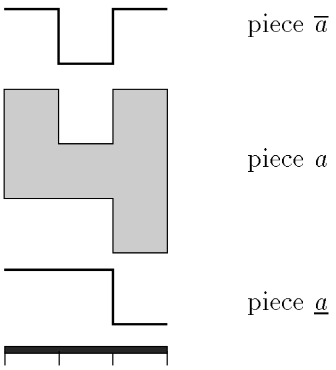

Fig. 5. A piece and the associated upper and lower contour pieces.

Given a vector $x \in \mathbb{R}_{\max}^{\mathcal{R}}$, interpreted as the upper contour of a heap, we define the *completed contour* $\phi(x) \in \mathbb{R}_{\max}^{\mathcal{R}}$ as follows

$$\phi(x)_i = \min\left(|x|_\oplus,\ \min_{a|i\in R(a)} x\mathcal{M}(\underline{a})_i\right) . \tag{18}$$

The vector $\phi(x)$ can be loosely described as the maximal upper contour such that the height of a heap piled up on x is the same as the height of a heap piled up on $\phi(x)$. More precisely, we have

$$\forall w \in \mathcal{A}^*,\ \ \phi(x)\mathcal{M}(w)\mathbb{1} = x\mathcal{M}(w)\mathbb{1} . \tag{19}$$

For the sake of completeness, let us prove (19). Given a word $w = w_1 \cdots w_n$, we define

$$R(w) = R(w_1) \cup \cdots \cup R(w_n) . \tag{20}$$

We are going to prove the following results which put together imply (19)

$$\begin{aligned} &\forall i \in R(w), && \phi(x)\mathcal{M}(w)_i = x\mathcal{M}(w)_i && (21)\\ &\forall i \notin R(w), && x\mathcal{M}(w)_i \le \phi(x)\mathcal{M}(w)_i \le |x\mathcal{M}(w)|_\oplus . && (22) \end{aligned}$$

It follows from the definition that (21) and (22) hold for the empty word e (setting $R(e) = \emptyset$). Assume now that (21) and (22) hold for all words of length less or equal than n. We consider the word wa where w is of length n and a is a letter.

If $i \notin R(a)$ and $i \in R(w)$, then

$$\phi(x)\mathcal{M}(w)\mathcal{M}(a)_i = \phi(x)\mathcal{M}(w)_i = x\mathcal{M}(w)_i = x\mathcal{M}(w)\mathcal{M}(a)_i .$$

If $i \notin R(a)$ and $i \notin R(w)$, then

$$\begin{aligned} x\mathcal{M}(w)\mathcal{M}(a)_i = x_i &\le \phi(x)_i = \phi(x)\mathcal{M}(w)\mathcal{M}(a)_i \\ &\le |x|_\oplus \le |x\mathcal{M}(w)\mathcal{M}(a)|_\oplus . \end{aligned}$$

If $i \in R(a)$, then

$$
\begin{aligned}
\phi(x)\mathcal{M}(w)\mathcal{M}(a)_i \;&=\; \bigoplus_{j\in R(a)} \phi(x)\mathcal{M}(w)_j\mathcal{M}(a)_{ji} \\
&= \bigoplus_{j\in R(a)\cap R(w)} \phi(x)\mathcal{M}(w)_j\mathcal{M}(a)_{ji} \oplus \bigoplus_{j\in R(a), j\notin R(w)} \phi(x)\mathcal{M}(w)_j\mathcal{M}(a)_{ji} \\
&= \bigoplus_{j\in R(a)\cap R(w)} x\mathcal{M}(w)_j\mathcal{M}(a)_{ji} \oplus \bigoplus_{j\in R(a), j\notin R(w)} \phi(x)_j\mathcal{M}(a)_{ji} \\
&\le \bigoplus_{j\in R(a)\cap R(w)} x\mathcal{M}(w)_j\mathcal{M}(a)_{ji} \oplus \phi(x)\mathcal{M}(a)_i \\
&= \bigoplus_{j\in R(a)\cap R(w)} x\mathcal{M}(w)_j\mathcal{M}(a)_{ji} \oplus x\mathcal{M}(a)_i \;=\; x\mathcal{M}(w)\mathcal{M}(a)_i\,.
\end{aligned}
$$

Since obviously $\phi(x)\mathcal{M}(w)\mathcal{M}(a)_i \geq x\mathcal{M}(w)\mathcal{M}(a)_i$, we get that $\phi(x)\mathcal{M}(w)\mathcal{M}(a)_i = x\mathcal{M}(w)\mathcal{M}(a)_i$. This concludes the proof of (21) and (22), hence of (19).

Given a contour $x \in \mathbb{R}_{\max}^{\mathcal{R}}$, we define the *normalized completed* contour $\varphi(x) = \pi(\phi(x))$. Let us define

$$
\varphi(\mathcal{H}) = \{\varphi(I\mathcal{M}(w)), w \in \mathcal{A}^*\}\,. \tag{23}
$$

Let us assume that $\varphi(\mathcal{H})$ is finite. Then we define the *contour-completed automaton* of $\mathcal{H}$. It is a deterministic, complete and trim (max,+) automaton over the alphabet $\mathcal{A}$, of dimension $\varphi(\mathcal{H})$. It is defined by $(\delta, \nu, \mathbb{1})$ where for $x, y \in \varphi(\mathcal{H}), a \in \mathcal{A}$,

$$
\delta_x = \begin{cases} |\phi(I)|_\oplus & \text{if } x = \varphi(I) \\ \mathbb{0} & \text{otherwise} \end{cases}, \quad \nu(a)_{xy} = \begin{cases} |\phi(x\mathcal{M}(a))|_\oplus & \text{if } \varphi(x\mathcal{M}(a)) = y \\ \mathbb{0} & \text{otherwise} \end{cases}.
$$

The automaton $\mathcal{H}$ and its contour-completed automaton recognize the same map, i.e.

$$
\forall w \in \mathcal{A}^*, \quad I\mathcal{M}(w)\mathbb{1} = \delta\nu(w)\mathbb{1}\,.
$$

The proof is analogous to the one of (17). The contour-completed automaton is used several times in §7, see for instance Example 16.

5.3 Piece-completed heap automaton

After having defined the completion of contours, we introduce in this section the completion of pieces.

We define the *upper-completed pieces* $a^\circ, a \in \mathcal{A}$, and the *lower-completed pieces* $a_\circ, a \in \mathcal{A}$, as follows: $R(a^\circ) = R(a_\circ) = R(a)$ and

$$l(a^\circ) = l(a), \quad \forall i \in R(a),\ u(a^\circ)_i = \min_{x | i \in R(x)} \max_{j \in R(x)} (u(a)_j + l(x)_i - l(x)_j) \quad (24)$$

$$u(a_\circ) = u(a), \quad \forall i \in R(a),\ l(a_\circ)_i = \max_{x | i \in R(x)} \min_{j \in R(x)} (l(a)_j + u(x)_i - u(x)_j)\,. \quad (25)$$

We check easily that $u(a^\circ) \geq l(a^\circ)$ and $u(a_\circ) \geq l(a_\circ)$, hence we have indeed defined pieces. Let us comment on this definition. Let x be a piece such that $R(x) \cap R(a) \neq \emptyset$. Let a' be the piece obtained by piling up a and the part of the lower contour piece $\underline{x}$ corresponding to the slots $R(x) \cap R(a)$. The piece a' is such that the heaps $a'x$ and ax are identical. Hence, the piece a° can be interpreted as the piece with lower contour $l(a)$ and with the largest possible upper contour such that the asymptotic behavior of a heap is not modified when replacing the occurrences of a by a°. There is an analogous interpretation for the pieces $a_\circ$. An illustration of upper and lower completion is given in Example 8 and Figure 6.

With the heap automaton $\mathcal{H} = (I, \mathcal{M}, \mathbb{1})$, we associate the heap automaton $\mathcal{H}^\circ = (I, \mathcal{M}^\circ, \mathbb{1})$ defined by $\mathcal{M}^\circ(a) = \mathcal{M}(a^\circ)$, and the heap automaton $\mathcal{H}_\circ = (I, \mathcal{M}_\circ, \mathbb{1})$ defined by $\mathcal{M}_\circ(a) = \mathcal{M}(a_\circ)$.

Lemma 7 *A heap automaton $\mathcal{H}$ is finitely distant from both the heap automaton $\mathcal{H}^\circ$ and the heap automaton $\mathcal{H}_\circ$.*

PROOF. Let us set

$$K^\circ = \bigoplus_{a \in \mathcal{A}} \bigoplus_{i,j \in R(a)} \mathcal{M}(a^\circ)_{ij} - \mathcal{M}(a)_{ij},\ K_\circ = \bigoplus_{a \in \mathcal{A}} \bigoplus_{i,j \in R(a)} \mathcal{M}(a_\circ)_{ij} - \mathcal{M}(a)_{ij}\,.$$

We want to prove the following inequalities, for all $w \in \mathcal{A}^*$,

$$\mathbb{1} \leq\ I\mathcal{M}^\circ(w)\mathbb{1} - I\mathcal{M}(w)\mathbb{1} \leq\ K^\circ \quad (26)$$

$$\mathbb{1} \leq\ I\mathcal{M}_\circ(w)\mathbb{1} - I\mathcal{M}(w)\mathbb{1} \leq\ K_\circ\,. \quad (27)$$

Since we have $\forall i,j,\ \mathcal{M}^\circ(a)_{ij} \geq \mathcal{M}(a)_{ij},\ \mathcal{M}_\circ(a)_{ij} \geq \mathcal{M}(a)_{ij}$, the left-hand side inequalities in (26) and (27) follow immediately. Let us prove the right-hand side inequality in (26), the proof of the one in (27) being similar.

First of all, for two words x and y over the alphabet $\mathcal{A}$, we have (where $R(x)$ and $R(y)$ are defined as in (20))

$$R(x) \cap R(y) = \emptyset \implies \mathcal{M}(x)\mathcal{M}(y) = \mathcal{M}(y)\mathcal{M}(x) = \mathcal{M}(x) \oplus \mathcal{M}(y)\,. \quad (28)$$

To prove (28), it is enough to remark that it follows from the definition in (4) that: $\forall x \in \mathcal{A}^*, \forall i \notin R(x), \mathcal{M}(x)_{ii} = \mathbb{1},\ \forall i,j \notin R(x), i \neq j, \mathcal{M}(x)_{ij} = \mathbb{0}$.

We need another intermediary result: for any two pieces $a, b \in \mathcal{A}$, we have

$$\forall i \in R(a), \forall j \in R(b), \quad \mathcal{M}(a^\circ b^\circ)_{ij} = \mathcal{M}(ab^\circ)_{ij} \ . \tag{29}$$

If $R(a) \cap R(b) = \emptyset$, then $\mathcal{M}(a^\circ b^\circ)_{ij} = \mathcal{M}(ab^\circ)_{ij} = \mathbb{0}$. Otherwise we have

$$\begin{aligned}
\mathcal{M}(a^\circ b^\circ)_{ij} &= \bigoplus_{k \in R(a) \cap R(b)} \mathcal{M}(a^\circ)_{ik} \mathcal{M}(b^\circ)_{kj} \\
&= \bigoplus_{k \in R(a) \cap R(b)} u(a^\circ)_k - l(a^\circ)_i + u(b^\circ)_j - l(b^\circ)_k \\
&= \bigoplus_{k \in R(a) \cap R(b)} u(a^\circ)_k - l(a)_i + u(b^\circ)_j - l(b)_k \\
&\leq \bigoplus_{k \in R(a) \cap R(b)} \bigoplus_{l \in R(b)} (u(a)_l + l(b)_k - l(b)_l) - l(a)_i + u(b^\circ)_j - l(b)_k \\
&= \bigoplus_{l \in R(b)} u(a)_l - l(a)_i + u(b^\circ)_j - l(b^\circ)_l \\
&= \bigoplus_{l} \mathcal{M}(a)_{il} \mathcal{M}(b^\circ)_{lj} \ = \ \mathcal{M}(ab^\circ)_{ij} \ .
\end{aligned}$$

Furthermore, it is immediate that $\mathcal{M}(a^\circ b^\circ)_{ij} \geq \mathcal{M}(ab^\circ)_{ij}$. This concludes the proof of (29).

Obviously, the right inequality in (26) holds for words of length 1. Let us assume that it holds for all words of length n. Let $w = w_1 \cdots w_{n+1}$ be a word of length $n+1$. Assume there exists $i \in \{1, \cdots n\}$ such that $R(w_i) \cap R(w_{i+1}) = \emptyset$, then using (28), we get

$$\mathcal{M}(w_1 \cdots w_{n+1}) = \mathcal{M}(w_1 \cdots w_{i-1} w_{i+1} \cdots w_{n+1}) \oplus \mathcal{M}(w_1 \cdots w_i w_{i+2} \cdots w_{n+1}) \ ,$$

with an analogous equality for $\mathcal{M}^\circ$. Setting $u = w_1 \cdots w_{i-1} w_{i+1} \cdots w_{n+1}$ and $v = w_1 \cdots w_i w_{i+2} \cdots w_{n+1}$, we deduce that we have

$$\begin{aligned}
I\mathcal{M}^\circ(w)\mathbb{1} - I\mathcal{M}(w)\mathbb{1} &= I\mathcal{M}^\circ(u)\mathbb{1} \oplus I\mathcal{M}^\circ(v)\mathbb{1} - I\mathcal{M}(u)\mathbb{1} \oplus I\mathcal{M}(v)\mathbb{1} \\
&\leq (I\mathcal{M}^\circ(u)\mathbb{1} - I\mathcal{M}(u)\mathbb{1}) \oplus (I\mathcal{M}^\circ(v)\mathbb{1} - I\mathcal{M}(v)\mathbb{1}) \\
&\leq K^\circ \ ,
\end{aligned}$$

where the last inequality is obtained by applying the recurrence assumption to the words u and v which are of length n. Assume now that $R(w_i) \cap R(w_{i+1}) \neq \emptyset$ for all $i \in \{1, \ldots, n\}$. Let j be such that $I\mathcal{M}^\circ(w)\mathbb{1} = I\mathcal{M}^\circ(w)_j$. Assume that $j \notin R(w_{n+1})$, then $I\mathcal{M}^\circ(w)\mathbb{1} = I\mathcal{M}^\circ(w_1 \cdots w_n)\mathbb{1}$ and

$$\begin{aligned}
I\mathcal{M}^\circ(w)\mathbb{1} - I\mathcal{M}(w)\mathbb{1} &= I\mathcal{M}^\circ(w_1 \cdots w_n)\mathbb{1} - I\mathcal{M}(w)\mathbb{1} \\
&\leq I\mathcal{M}^\circ(w_1 \cdots w_n)\mathbb{1} - I\mathcal{M}(w_1 \cdots w_n)\mathbb{1} \leq K^\circ \ .
\end{aligned}$$

The case $j \in R(w_{n+1})$ remains to be treated. We obtain, using recursively (29), that

$$I\mathcal{M}^\circ(w)_j = I\mathcal{M}(w_1^\circ \cdots w_{n+1}^\circ)_j = I\mathcal{M}(w_1 \cdots w_n w_{n+1}^\circ)_j \,.$$

We conclude that

$$I\mathcal{M}^\circ(w)\mathbb{1} - I\mathcal{M}(w)\mathbb{1} \leq I\mathcal{M}(w_1 \cdots w_n w_{n+1}^\circ)_j - I\mathcal{M}(w_1 \cdots w_n w_{n+1})_j \leq K^\circ \,,$$

by definition of K°. This completes the proof.

We define the *bi-completed* pieces $a_\circ^\circ, a \in \mathcal{A}$, as follows: $R(a_\circ^\circ) = R(a)$ and

$$l(a_\circ^\circ) = l(a_\circ), \quad \forall i \in R(a),\ u(a_\circ^\circ)_i = \min_{x | i \in R(x)} \max_{j \in R(x)} (u(a_\circ)_j + l(x_\circ)_i - l(x_\circ)_j) \,.$$

Here the pieces $a_\circ^\circ, a \in \mathcal{A}$, are obtained by lower-completion first and then upper-completion. We can also define pieces, say $\hat{a}_\circ^\circ, a \in \mathcal{A}$, by performing upper-completion first and then lower-completion, that is: $R(\hat{a}_\circ^\circ) = R(a)$ and

$$u(\hat{a}_\circ^\circ) = u(a^\circ), \quad \forall i \in R(a),\ l(\hat{a}_\circ^\circ)_i = \max_{x | i \in R(x)} \min_{j \in R(x)} (l(a^\circ)_j + u(x^\circ)_i - u(x^\circ)_j) \,.$$

In general, the pieces $a_\circ^\circ$ and $\hat{a}_\circ^\circ$ are different, in other words the operations of upper and lower-completion do not commute. An example of bi-completion is provided in Figure 6. On this example, the pieces $a_\circ^\circ$ and $\hat{a}_\circ^\circ$ (resp. $b_\circ^\circ$ and $\hat{b}_\circ^\circ$) are different.

Example 8 *Consider the heap automaton with pieces defined by*

$$l(a) = (\mathbb{1}, \mathbb{1}),\ u(a) = (1, 3),\ \text{and } l(b) = (1, \mathbb{1}),\ u(b) = (2, 3) \,.$$

It is simpler to obtain the completed pieces graphically, using the intuition described above. We have represented in Figure 6 the upper, lower and bi-completed pieces: $\{a^\circ, b^\circ\}, \{a_\circ, b_\circ\}, \{a_\circ^\circ, b_\circ^\circ\}$ *and* $\{\hat{a}_\circ^\circ, \hat{b}_\circ^\circ\}$.

The heap automaton $\mathcal{H}_\circ^\circ = (I, \mathcal{M}_\circ^\circ, \mathbb{1})$, over the alphabet $\mathcal{A}$, defined by $\mathcal{M}_\circ^\circ(a) = \mathcal{M}(a_\circ^\circ)$, is called the *piece-completed* heap automaton associated with $\mathcal{H} = (I, \mathcal{M}, \mathbb{1})$.

Lemma 9 *A heap automaton $\mathcal{H}$ and the associated piece-completed automaton $\mathcal{H}_\circ^\circ$ are finitely distant.*

PROOF. By definition, we have $\mathcal{H}_\circ^\circ = (\mathcal{H}_\circ)^\circ$. By applying Lemma 7 twice, we get the result.

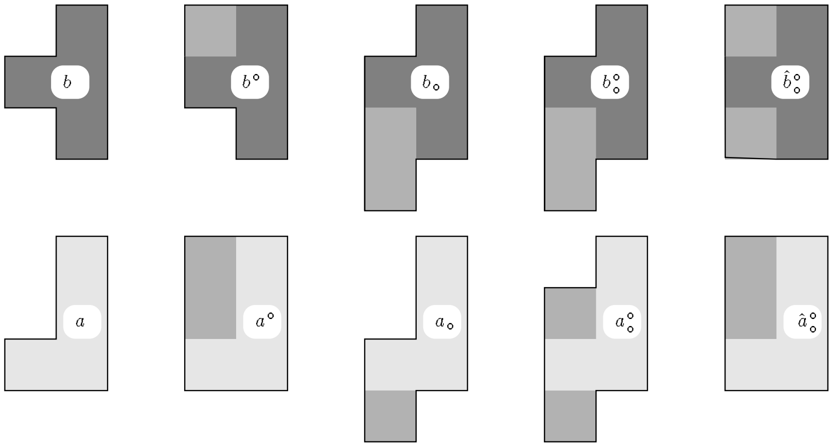

Fig. 6. Two pieces and the associated upper-completed, lower-completed and bi-completed pieces.

Given a set of pieces $\mathcal{A}$, let us denote by $\mathcal{A}^\circ, \mathcal{A}_\circ$, and $\mathcal{A}_\circ^\circ$ the upper-completed, lower-completed and bi-completed sets of pieces. Given two pieces a and b, we say that r is a *contact slot* for ab if $\mathcal{M}(ab)_{ij} = \mathcal{M}(a)_{ir}\mathcal{M}(b)_{rj}, \forall i \in R(a), j \in R(b)$ (visually, a is in contact with b at slot r in the heap ab).

Lemma 10 *We have $(\mathcal{A}^\circ)^\circ = \mathcal{A}^\circ$, $(\mathcal{A}_\circ)_\circ = \mathcal{A}_\circ$ and $(\mathcal{A}_\circ^\circ)_\circ^\circ = \mathcal{A}_\circ^\circ$. In words, a set of lower-completed (resp. upper-completed or bi-completed) pieces is left unchanged by performing another lower (resp upper or bi) completion.*

PROOF. The arguments below are based on the following immediate remark: Given a and b in the same set of pieces, if i is a contact slot of ab then $l(b_\circ)_i = l(b)_i$ and $u(a^\circ)_i = u(a)_i$.

By definition we have, $\forall a \in \mathcal{A}, \forall i \in R(a), \exists b \in \mathcal{A}, i \in R(b), \exists j(i) \in R(b)$,

$$l(a_\circ)_i = l(a)_{j(i)} + u(b)_i - u(b)_{j(i)} \, .$$

It implies that $j(i)$ is a contact slot for ba and that both i and $j(i)$ are contact slots for $ba_\circ$. Obviously, it implies that i is a contact slot for $b_\circ a_\circ$ and we conclude that $l((a_\circ)_\circ)_i = l(a_\circ)_i$. This completes the proof of $(\mathcal{A}_\circ)_\circ = \mathcal{A}_\circ$. The proof of $(\mathcal{A}^\circ)^\circ = \mathcal{A}^\circ$ is similar.

Since i is a contact slot for $b_\circ a_\circ$, we also obtain that $u(b_\circ^\circ)_i = u(b_\circ)_i$. Hence, for all k, we have $\mathcal{M}(b_\circ^\circ)_{ki} = \mathcal{M}(b_\circ)_{ki}$. We also have that i is a contact slot for $b_\circ a_\circ^\circ$. Using this together with (29), we get that $\forall k \in R(b), \forall l \in R(a)$,

$$\mathcal{M}(b_\circ^\circ a_\circ^\circ)_{kl} = \mathcal{M}(b_\circ a_\circ^\circ)_{kl} = \mathcal{M}(b_\circ)_{ki}\mathcal{M}(a_\circ^\circ)_{il} = \mathcal{M}(b_\circ^\circ)_{ki}\mathcal{M}(a_\circ^\circ)_{il} \, .$$

It implies that $l((a_\circ^\circ)_\circ)_i = l(a_\circ^\circ)_i$. We deduce that we have $(a_\circ^\circ)_\circ = a_\circ^\circ$ and we can prove in a similar way that $(a_\circ^\circ)^\circ = a_\circ^\circ$. We conclude that $(\mathcal{A}_\circ^\circ)_\circ^\circ = \mathcal{A}_\circ^\circ$.

Both the contour completion of §5.2 and the above piece completion are based on the idea of local transformations which do not modify the asymptotic behavior of heaps. However, they are different: the completed contours are not the upper contours of the heaps of completed pieces.

6 Minimal Realization

The goal of this section is to prove that given a heap automaton with two pieces, there exists a finitely distant one of dimension at most 3, Theorem 12.

A set of *bi-complete* pieces is a set $\mathcal{A}$ such that $\mathcal{A}_\circ^\circ = \mathcal{A}$. From now on, we always implicitly consider bi-complete pieces. Due to Lemma 9 and 10, we can make this assumption without loss of generality.

Let $\mathcal{H} = (I, \mathcal{M}, \mathbb{1})$ be a heap automaton with set of slots $\mathcal{R}$ and let $\tilde{\mathcal{R}}$ be a subset of $\mathcal{R}$. The heap model obtained by restriction of $\mathcal{H}$ to $\tilde{\mathcal{R}}$ is denoted by $\mathcal{H}_{|\tilde{\mathcal{R}}}$ and defined by $\mathcal{H}_{|\tilde{\mathcal{R}}} = (I_{|\tilde{\mathcal{R}}}, \mathcal{M}_{|\tilde{\mathcal{R}}\times\tilde{\mathcal{R}}}, \mathbb{1})$ (visually, the new pieces are the old ones restricted to $\tilde{\mathcal{R}}$).

Lemma 11 *Let $\mathcal{H}$ be a heap automaton on the alphabet $\mathcal{A}$ and with set of slots $\mathcal{R}$. Let $\tilde{\mathcal{R}}$ be a subset of $\mathcal{R}$. The automaton $\mathcal{H}_{|\tilde{\mathcal{R}}}$ is finitely distant from $\mathcal{H}$ if and only if $\tilde{\mathcal{R}}$ contains a contact slot for each word $ab, a, b \in \mathcal{A}$, such that $R(a) \cap R(b) \neq \emptyset$.*

PROOF. Let $\mathcal{H} = (I, \mathcal{M}, \mathbb{1})$. Assume that $\tilde{\mathcal{R}}$ contains at least one contact slot for each ab such that $R(a) \cap R(b) \neq \emptyset$. Let (a, b) be such a couple. We have, by definition of a contact slot,

$$\mathcal{M}(ab)_{|\tilde{\mathcal{R}}\times\tilde{\mathcal{R}}} = \mathcal{M}(a)_{|\tilde{\mathcal{R}}\times\tilde{\mathcal{R}}}\mathcal{M}(b)_{|\tilde{\mathcal{R}}\times\tilde{\mathcal{R}}} = \mathcal{M}_{|\tilde{\mathcal{R}}\times\tilde{\mathcal{R}}}(ab)\,. \tag{30}$$

Let us consider a word $w \in \mathcal{A}^*$. Using repeatedly the equality in (28), we obtain that $\mathcal{M}(w) = \bigoplus_{v\in\mathcal{I}(w)} \mathcal{M}(v)$, where v belongs to $\mathcal{I}(w)$ if v is a subword of w and if two consecutive letters of v, say v_i and v_{i+1}, are such that $R(v_i) \cap R(v_{i+1}) \neq \emptyset$. For each word $v \in \mathcal{I}(w)$, we obtain by using repeatedly (30) that $\mathcal{M}(v)_{|\tilde{\mathcal{R}}\times\tilde{\mathcal{R}}} = \mathcal{M}_{|\tilde{\mathcal{R}}\times\tilde{\mathcal{R}}}(v)$. We deduce that $\mathcal{M}(w)_{|\tilde{\mathcal{R}}\times\tilde{\mathcal{R}}} = \mathcal{M}_{|\tilde{\mathcal{R}}\times\tilde{\mathcal{R}}}(w)$. We conclude easily that

$$\begin{aligned}\mathbb{1} \leq & \sup_{w\in\mathcal{A}^*} \left\{ I\mathcal{M}(w)\mathbb{1} - I_{|\tilde{\mathcal{R}}}\mathcal{M}_{|\tilde{\mathcal{R}}\times\tilde{\mathcal{R}}}(w)\mathbb{1} \right\} \\ = & \sup_{w\in\mathcal{A}^*} \left\{ I\mathcal{M}(w)\mathbb{1} - I\mathcal{M}(w)_{|\tilde{\mathcal{R}}}\mathbb{1} \right\} \;\leq\; |I|_\oplus \otimes [\bigoplus_{a\in\mathcal{A}} |\mathcal{M}(a)|_\oplus]^2\,.\end{aligned}$$

Hence, $\mathcal{H}_{|\tilde{\mathcal{R}}}$ is finitely distant from $\mathcal{H}$. We have shown that the condition is sufficient. Let us prove that it is necessary. Assume that ab, $R(a) \cap R(b) \neq \emptyset$, has no contact slot in $\tilde{\mathcal{R}}$. Let δ be the minimal gap between a and b in the heap ab over the slots $\tilde{\mathcal{R}}$. Then we have $|\mathcal{M}(ab)|_{\oplus} - |\mathcal{M}_{|\tilde{\mathcal{R}}\times\tilde{\mathcal{R}}}(ab)|_{\oplus} = \delta > 0$. It implies that $|\mathcal{M}((ab)^n)|_{\oplus} - |\mathcal{M}_{|\tilde{\mathcal{R}}\times\tilde{\mathcal{R}}}((ab)^n)|_{\oplus} \geq n \times \delta$, showing that $\mathcal{H}$ and $\mathcal{H}_{|\tilde{\mathcal{R}}}$ are not finitely distant.

Theorem 12 *Let $\mathcal{H} = (I, \mathcal{M}, \mathbb{1})$ be a heap automaton with two pieces. Over the same alphabet, there exists a heap automaton $\tilde{\mathcal{H}} = (\tilde{I}, \tilde{\mathcal{M}}, \mathbb{1})$ of dimension at most 3 and which is finitely distant from $\mathcal{H}$.*

PROOF. By choosing one contact slot for each one of the words aa, ab, ba and bb, we obtain a set $\tilde{\mathcal{R}}$ of cardinality at most 4 and such that the automaton $\mathcal{H}_{|\tilde{\mathcal{R}}}$ is finitely distant from $\mathcal{H}$, see Lemma 11. We now prove that 3 slots are always enough. We define the application $c : \mathcal{R} \to \mathcal{P}(\mathcal{A}^2)$, where $\mathcal{P}(\mathcal{A}^2)$ denotes the set of subsets of $\mathcal{A}^2$. The set $c(r)$ contains xy if r is a contact slot of xy. Assume that $R(a) \cap R(b) \neq \emptyset$ and consider a slot $r \in R(a) \cap R(b)$. Let us prove that $c(r)$ must contain words starting with a and b and words finishing with a and b. Assume for instance that $c(r)$ does not contain any word starting with a. Then, according to (24), there exists $x \in \mathcal{A}$ such that

$$u(a^\circ)_r = \max_{j \in R(x)} u(a)_j + l(x)_r - l(x)_j \,.$$

Since ax does not belong to $c(r)$, the maximum above is attained for $j \neq r$ and we have $u(a^\circ)_r > u(a)_r$. This contradicts the fact that $\mathcal{A}$ is a set of bi-complete pieces.

To summarize, we must have

$$\{aa, bb\} \subset c(r) \ \text{ or } \{ab, ba\} \subset c(r) \,. \tag{31}$$

If we have $\{aa, bb\} \subset c(r)$ (resp. $\{ab, ba\} \subset c(r)$), we complete the slot r with a contact slot for the heap ab and one for the heap ba (resp. for aa and bb). We have a set of at most 3 slots which satisfies the required properties.

Now assume that $R(a) \cap R(b) = \emptyset$. It is enough for $\tilde{\mathcal{R}}$ to contain a contact slot of aa and one of bb, hence to be of cardinality 2, for $\mathcal{H}_{|\tilde{\mathcal{R}}}$ to be finitely distant from $\mathcal{H}$. This completes the proof.

Performed on the original heap automaton, instead of the piece-completed one, the above argument would not work. Consider the heap model $\mathcal{H}$ of dimension 4 defined by $l(a) = (\mathbb{1}, \mathbb{1}, 1, \mathbb{0}), u(a) = (3, 2, 3, \mathbb{0}), l(b) = (\mathbb{0}, 1, \mathbb{1}, \mathbb{1})$ and $u(b) = (\mathbb{0}, 3, 2, 3)$. There exists no proper subset $\tilde{\mathcal{R}}$ of $\mathcal{R}$ such that $\mathcal{H}_{|\tilde{\mathcal{R}}}$ is finitely distant from $\mathcal{H}$.

Example 13 *Let us illustrate Theorem 12. We consider the heap automaton* $\mathcal{H} = (\mathbb{1}, \mathcal{M}, \mathbb{1})$ *of dimension 4 and consisting of the two bi-complete pieces defined by*

$$l(a) = (\mathbb{1}, 3, 2, \mathbb{0}),\ u(a) = (4, 4, 5, \mathbb{0}),\ \text{and}\ l(b) = (\mathbb{0}, 2, 3, \mathbb{1}),\ u(b) = (\mathbb{0}, 5, 4, 4)\ .$$

We have $c(1) = \{aa\}, c(2) = \{ab, ba\}, c(3) = \{ab, ba\}$ *and* $c(4) = \{bb\}$. *Here, we can choose either* $\tilde{\mathcal{R}} = \{1, 2, 4\}$ *or* $\{1, 3, 4\}$ *and the heap automaton* $\mathcal{H}_{|\tilde{\mathcal{R}}}$ *will be finitely distant from* $\mathcal{H}$. *This can be 'checked' on Figure 7. In this*

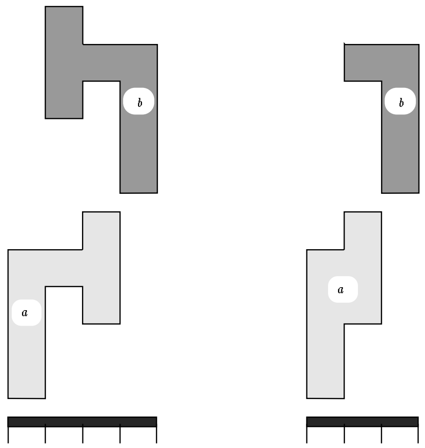

Fig. 7. A heap automaton of dimension 4 and a finitely distant one of dimension 3.

example, we do not always have $\mathbb{1}\mathcal{M}(w)\mathbb{1} = \mathbb{1}\mathcal{M}_{|\tilde{\mathcal{R}}\times\tilde{\mathcal{R}}}(w)\mathbb{1}$. *However we can check that* $\mathbb{1} \leq \mathbb{1}\mathcal{M}(w)\mathbb{1} - \mathbb{1}\mathcal{M}_{|\tilde{\mathcal{R}}\times\tilde{\mathcal{R}}}(w)\mathbb{1} \leq 1$.

Lemma 11 and Theorem 12 are *minimal realization* type of results. Here is the generic problem of this kind: *Given an automaton with multiplicities in a semiring, find another automaton recognizing the same map and of minimal dimension.*

In a commutative field, the minimal realization problem is solved, see [8] for a proof and references. In $\mathbb{R}_{\max}$, it is a well-known difficult and unsolved problem, see [18] for partial results and references. Here, our result is specific in several ways. First, we look at a particular type of (max,+) automata, heap automata with two pieces. Second, we look for a realization by a heap automaton and not by an arbitrary (max,+) automaton. Third, we only require an approximate type of realization, see (15).

6.1 Classification of heap models with two pieces

As a by-product of Theorem 12, to study heap automata with two pieces, it is enough to consider automata with bi-complete pieces and of dimension at most 3. We are going to show that there are only four cases which need to be treated (up to a renaming of pieces and slots) which are:

$$\begin{array}{ll} \mathcal{H} = (\{a,b\},\{1,2\},R,u,l,I) & R(a)=\{1\}, R(b)=\{2\} \\ & R(a)=\{1,2\}, R(b)=\{1,2\} \\ & R(a)=\{1,2\}, R(b)=\{2\} \\ \mathcal{H} = (\{a,b\},\{1,2,3\},R,u,l,I) & R(a)=\{1,2\}, R(b)=\{2,3\}\,. \end{array}$$

We recall that the function $c(.)$ was defined in the proof of Theorem 12.

(i) If $R(a)\cap R(b)=\emptyset$, we have seen in the proof of Theorem 12, that the heap model can be represented with two slots only, one for each piece.

(ii) Let us assume that $R(a)=R(b)$. Let r be such that $aa\in c(r)$. Using (31), we have either $\{aa,bb\}\subset c(r)$ or $\{aa,ab,ba\}\subset c(r)$. If we are in the second case, we complete r with a contact slot for bb. If we are in the first case, let us consider a slot r' such that $ab\in c(r')$. We have, as before, either $\{ab,ba\}\subset c(r')$ or $\{ab,aa,bb\}\subset c(r')$. If $\{ab,ba\}\subset c(r')$, then we select the slots $\{r,r'\}$. If $\{ab,aa,bb\}\subset c(r')$, then we complete r' with a contact slot for ba. In all cases, we obtain a finitely distant heap model with at most two slots.

(iii) Let us assume that $R(b)\subset R(a), R(b)\neq R(a)$. Let r be a slot such that $bb\in c(r)$. Since $r\in R(a)\cap R(b)$, we must have either $\{bb,ab,ba\}\subset c(r)$ or $\{aa,bb\}\subset c(r)$. In the second case, we conclude as in (ii). In the first case, we complete r with a slot r' such that $aa\in c(r')$. Compared with (ii), there is a new possible situation: two slots $\{r,r'\}$ with $R(a)=\{r,r'\}$ and $R(b)=\{r\}$.

(iv) Let us assume that $R(a)\cap R(b)\neq\emptyset, R(a)\backslash R(b)\neq\emptyset, R(b)\backslash R(a)\neq\emptyset$. We consider a slot $r\in R(a)\cap R(b)$ and such that $ab\in c(r)$. We have either $\{ab,aa,bb\}\subset c(r)$ or $\{ab,ba\}\subset c(r)$. In the first case, we complete r with a slot r' such that $ba\in c(r')$. In the second case, we complete r with a contact slot r_a for aa and a contact slot r_b for bb. Compared with the cases (ii) and (iii), there is a new possible situation: three slots $\{r,r_a,r_b\}$ with $R(a)=\{r,r_a\}$ and $R(b)=\{r,r_b\}$.

7 Heap Models with Two Pieces: Optimal Case

Let $\mathcal{H}$ be a heap model with two pieces. To solve the optimal problem, it is sufficient to consider the typical cases described in §6.1. Two situations need to be distinguished:

- $\mathcal{H}$ is 'determinizable', i.e. there exists a finitely distant, trim, and determin-

istic (max,+) automaton;
- $\mathcal{H}$ is 'not-determinizable'.

For 'determinizable' automata, there exists a periodic optimal schedule. We will see below that there are two cases where $\mathcal{H}$ is 'not-determinizable'. In both cases, we are able to identify 'visually' the optimal schedules. The resulting theorem can be stated as follows.

Theorem 14 *Let us consider a heap model with two pieces. There exists an optimal schedule which is balanced, either periodic or Sturmian.*

PROOF. We consider in §7.1-7.4 the four different cases described in §6.1. For each case, we prove that the results of Theorem 14 hold. Furthermore we provide an explicit way to compute $\rho_{\min}(\mathcal{H})$ and an optimal schedule in each case.

In the sections below, we always denote the heap model considered by $\mathcal{H} = (\mathcal{A}, \mathcal{R}, R, u, l, I)$ with $\mathcal{A} = \{a, b\}$ and $\mathcal{R} = \{1, 2\}$ or $\{1, 2, 3\}$. Viewed as a heap automaton, it is denoted by $\mathcal{H} = (I, \mathcal{M}, \mathbb{1})$. We always implicitly assume that we are working with bi-complete pieces. We recall that by modifying the ground shape in a heap automaton, we obtain a finitely distant automaton. Below we choose the ground shape which is the most adapted to each case.

If one of the two pieces, say a, satisfies $l(a) = u(a)$, then the optimal problem becomes trivial. We have $\rho_{\min}(\mathcal{H}) = \mathbb{1}$ and a periodic optimal schedule is provided by a^ω. From now on, we assume that $l(a) \neq u(a)$ and $l(b) \neq u(b)$. We set

$$h_a = \bigoplus_{i \in R(a)} u(a)_i - l(a)_i, \;\; h_b = \bigoplus_{i \in R(b)} u(b)_i - l(b)_i \, .$$

7.1 The case $R(a) = \{1\}, R(b) = \{2\}$

We assume that the ground shape is $\mathbb{1}$. We claim that the jump word u with characteristics $(h_a, h_b, 0)$ (see §4) is optimal. Furthermore, we have $\rho_{\min}(\mathcal{H}) = h_a h_b/(h_a + h_b)$. An example is provided in Figure 8.

We now prove these assertions. Let us pile up the pieces according to the jump word u defined by $(h_a, h_b, 0)$. We have, by construction, $\mid x_{\mathcal{H}}(u[n])_1 - x_{\mathcal{H}}(u[n])_2 \mid \leq \max(h_a, h_b)$. Hence we have $\lim_n x_{\mathcal{H}}(u[n])_1/n = \lim_n x_{\mathcal{H}}(u[n])_2/n$. Now, as the heap is without any gap, it implies immediately that u is optimal. The optimal schedule u is balanced, periodic when h_a/h_b is rational and Sturmian when h_a/h_b is irrational, see §4. We have

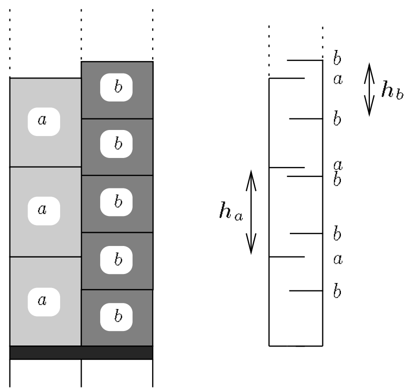

Fig. 8. The jump word $(h_a, h_b, 0)$ is optimal.

$$\rho_{\min}(\mathcal{H}) = \lim_n \frac{x_{\mathcal{H}}(u[n])_1}{n} = \lim_n \frac{h_a|u[n]|_a}{|u[n]|_a + |u[n]|_b} = \frac{h_a h_b}{h_a + h_b} .$$

To be complete, let us prove that it is not possible to find a periodic optimal schedule in the case h_a/h_b irrational. Let v be a finite word and let us consider the schedule v^ω. Since h_a/h_b is irrational, we have $h_a|v|_a \neq h_b|v|_b$. Let us assume that $h_a|v|_a > h_b|v|_b$. It implies that $|v|_a > |v|h_b/(h_a + h_b)$. We obtain

$$\lim_n y_{\mathcal{H}}(v^n)/|v^n| = \lim_n h_a|v^n|_a/|v^n| = h_a|v|_a/|v| > h_a h_b/(h_a + h_b) .$$

7.2 The case $R(a) = \{1, 2\}, R(b) = \{1, 2\}$

As $R(a) = R(b) = \mathcal{R}$, we have $\pi(x\mathcal{M}(a)) = \pi(y\mathcal{M}(a))$ and $\pi(x\mathcal{M}(b)) = \pi(y\mathcal{M}(b))$, for all $x, y \in \mathbb{R}^2$. Let us choose the ground shape to be $\pi(\mathbb{1}\mathcal{M}(a))$. We have $\pi(\mathcal{H}) = \{\pi(I\mathcal{M}(a)), \pi(I\mathcal{M}(b))\}$. Hence we can solve the optimal problem using the Cayley automaton, see §5.1. Applying the results of §3.2, it is always the case that one of the schedules a^ω, b^ω or $(ab)^\omega$ is optimal. These schedules are obviously balanced.

Example 15 *Consider the heap automaton $\mathcal{H}$ with pieces defined by*

$$l(a) = (1, \mathbb{1}),\ u(a) = (3, 2),\ \textit{and}\ l(b) = (\mathbb{1}, 1),\ u(b) = (2, 3) .$$

We have represented the pieces in Fig. 9. We check easily that $\pi(\mathcal{H}) = \{((\mathbb{1}, -1), (-1, \mathbb{1})\}$ (the ground shape being $(\mathbb{1}, -1)$). Let $(\alpha, \mu, \mathbb{1})$ be the Cayley automaton and let $M = \min(\mu(a), \mu(b))$. We have $\alpha = (\mathbb{1}, \mathbb{0})$ and

$$\mu(a) = \begin{pmatrix} 2 & \mathbb{0} \\ 3 & \mathbb{0} \end{pmatrix},\quad \mu(b) = \begin{pmatrix} \mathbb{0} & 3 \\ \mathbb{0} & 2 \end{pmatrix},\quad M = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} .$$

The minimal eigenvalue of the $\mathbb{R}_{\min}$ matrix M is 2, the circuits of minimal

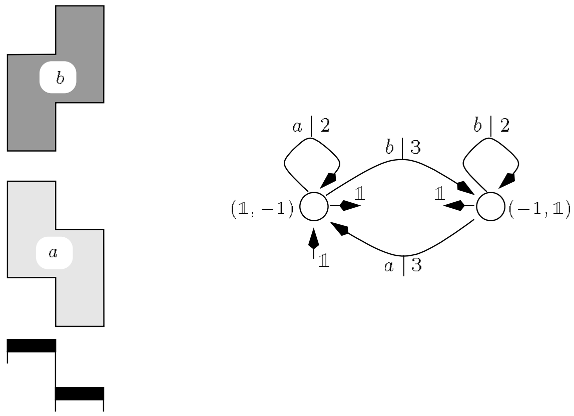

Fig. 9. Heap model with two pieces and its Cayley automaton.

mean weight are $\{2\}$ *and* $\{3\}$ *and* $M_{22} = \mu(a)_{22}, M_{33} = \mu(b)_{33}$. *We have* $\rho_{\min}(\mathcal{H}) = 2$ *and* a^ω *and* b^ω *are optimal schedules.*

7.3 The case $R(a) = \{1,2\}, R(b) = \{2\}$

This case could be reduced to the case $R(a) = \{1,2\}, R(b) = \{1,2\}$ (the one in §7.4) by adding a third slot and setting $R(b) = \{2,3\}$ and $u(b)_3 = u(b)_2$, $l(b)_3 = l(b)_2$. We treat the case $R(a) = \{1,2\}, R(b) = \{2\}$ separately in order to get more precise results. Let us set $\delta = l(a)_1 - l(a)_2$. For $u = (u_1, u_2) \in \mathbb{R}^2$, we obtain, see (18),

$$\begin{aligned}
\phi(u)_1 &= \min(u\mathcal{M}(\underline{a})_1, |u|_\oplus) = \min(u_1 \oplus \delta u_2, u_1 \oplus u_2) \\
\phi(u)_2 &= \min(u\mathcal{M}(\underline{a})_2, u\mathcal{M}(\underline{b})_2, |u|_\oplus) = u_2 .
\end{aligned}$$

Hence, we have

$$\varphi(u) = \begin{cases} (\mathbb{1}, \mathbb{1}) & \text{if } u_1 \le u_2 \le \delta u_2 \\ (\delta, \mathbb{1}) & \text{if } u_1 \le \delta u_2 \le u_2 \\ \pi(u_1, u_2) & \text{otherwise} \end{cases} . \qquad (32)$$

Assume that $\delta \ge 0$ and let the ground shape be equal to $\mathbb{1}\mathcal{M}(a)$. We have, $\forall u \in \mathbb{R}^2, \varphi(u\mathcal{M}(a)) = \varphi(\mathbb{1}\mathcal{M}(a))$. We deduce that

$$\varphi(\mathcal{H}) = \{\varphi(\mathbb{1}\mathcal{M}(ab^n)), n \in \mathbb{N}\} .$$

We also have $\mathbb{1}\mathcal{M}(ab^{n+1}) - \mathbb{1}\mathcal{M}(ab^n) = (\mathbb{1}, h_b)$. By assumption, we have $h_b > 0$. Hence there exists a smallest integer m such that

$$u(a)_2 + m \times h_b \ge u(a)_1 \iff \mathbb{1}\mathcal{M}(ab^m)_2 - \mathbb{1}\mathcal{M}(ab^m)_1 \ge 0 .$$

It implies, using (32), that $\forall n \geq m, \varphi(\mathbb{1}\mathcal{M}(ab^n)) = (\mathbb{1},\mathbb{1})$. We conclude that

$$\varphi(\mathcal{H}) = \{\varphi(\mathbb{1}\mathcal{M}(ab^n)), n \in \{0,\ldots,m\}\} \ .$$

We have proved that $\varphi(\mathcal{H})$ is finite. In the case $\delta \leq 0$, a similar analysis holds. In all cases we can solve the optimal problem using the contour-completed automaton and the results of §3.2. We have represented in Figure 10, the

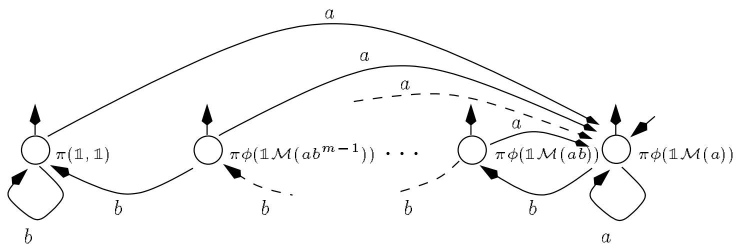

Fig. 10. Contour-completed automaton

contour-completed automaton in the case $m \geq 3$ and $\delta \geq 0$ (without the multiplicities). There are exactly $m+2$ simple circuits in this automaton with respective labels b and $ab^n, 0 \leq n \leq m$. For $0 \leq n \leq m-2$, the multiplicity to go from $\varphi(\mathbb{1}\mathcal{M}(ab^n))$ to $\varphi(\mathbb{1}\mathcal{M}(ab^{n+1}))$ is $\mathbb{1}$ while the one to go from $\varphi(\mathbb{1}\mathcal{M}(ab^{n+1}))$ to $\varphi(\mathbb{1}\mathcal{M}(a))$ is always equal to h_a. Hence the circuits of label $ab^n, 0 \leq n \leq m-2$, are not of minimal mean weight. We conclude that an optimal schedule can be found among the schedules $(ab^m)^\omega$,$(ab^{m-1})^\omega$ or b^ω (if $m = 0$, then either a^ω or b^ω is optimal). These schedules are balanced.

Example 16 *We consider the heap automaton with pieces a and b defined by*

$$l(a) = (\mathbb{1},\mathbb{1}), u(a) = (3,\mathbb{1}), \quad \textit{and} \quad l(b) = (\mathbb{0},\mathbb{1}), u(b) = (\mathbb{0},1) \ .$$

The pieces are represented in Figure 11. The completion operation has the

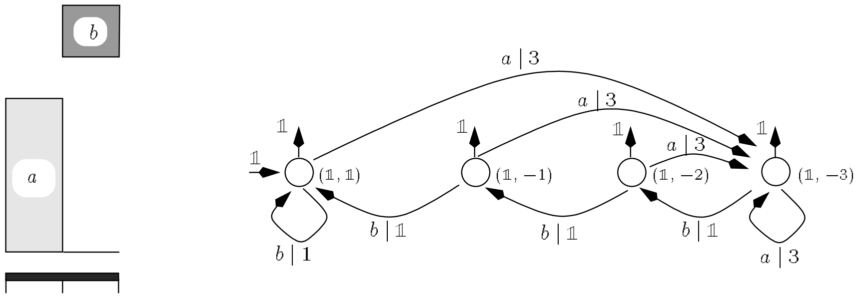

Fig. 11. Heap model and its contour-completed automaton.

following effect: $\phi(m,n) = (m,n)$ *if* $m \geq n$ *and* $\phi(m,n) = (n,n)$ *if* $m < n$. *Hence we have* $\varphi(\mathcal{H}) = \{(\mathbb{1},\mathbb{1}),(\mathbb{1},-1),(\mathbb{1},-2),(\mathbb{1},-3)\}$. *Let* $(\alpha,\mu,\mathbb{1})$ *be the contour-completed automaton and let* $M = \min(\mu(a),\mu(b))$. *The minimal eigenvalue of* M *is* $\rho_{\min}(M) = 3/4$ *and the circuit of minimal mean weight is labelled by abbb. We conclude that* $\rho_{\min}(\mathcal{H}) = 3/4$ *and that an optimal schedule is* $(abbb)^\omega$.

7.4 The case $R(a) = \{1, 2\}, R(b) = \{2, 3\}$

Two situations need to be considered: (i) the case $u(a)_2 = l(a)_2$ and $u(b)_2 = l(b)_2$; (ii) the case $u(a)_2 > l(a)_2$ (with the case $u(b)_2 > l(b)_2$ being treated similarly).

Case (i): $u(a)_2 = l(a)_2$ and $u(b)_2 = l(b)_2$

Assume that there exists an infinite heap w with an infinite number of each piece and without any 'gap' at slots 1 and 3. Now, we focus on the second slot of the heap w. The heights of the pieces a and b at slot 2 are given by $\{I_1 + l(a)_2 - l(a)_1 + nh_a, n \in \mathbb{N}\}$ and $\{I_3 + l(b)_2 - l(b)_3 + nh_b, n \in \mathbb{N}\}$ respectively. We set the ground shape to be

$$I = (h_a - l(a)_2 + l(a)_1, \mathbb{1}, h_b - l(b)_2 + l(b)_3) .$$

The heights of the pieces at slot 2 are now given by $\{nh_a, n \in \mathbb{N}^*\}$ and $\{nh_b, n \in \mathbb{N}^*\}$. Hence, the sequence of labels (read from bottom to top) at slot 2 is the jump word w defined by $(h_a, h_b, 0)$. Now, if we pile up the pieces according to w, we indeed obtain a heap without any gap on slots 1 and 3. An illustration is given in Figure 12. On slot 2, the pieces have been shortened to facilitate their identification. If h_a/h_b is rational

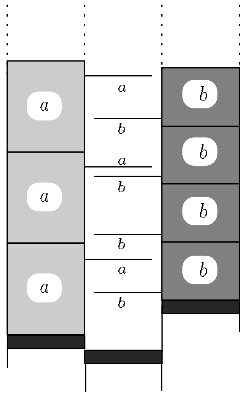

Fig. 12. The optimal heap is the jump word $babbaba \cdots$.

then w is balanced and periodic and otherwise it is Sturmian. If h_a/h_b is irrational, there does not exist any periodic optimal schedule. At last, we have $\rho_{\min}(\mathcal{H}) = h_a h_b/(h_a + h_b)$. The proof is exactly the same as in §7.1.

Case (ii): $u(a)_2 > l(a)_2$

Assume that $u(a)_2 - l(a)_2 > u(a)_1 - l(a)_1$. Then

$$u(a^\circ)_1 = \max(u(a)_1, u(a)_2 - l(a)_2 + l(a)_1) = u(a)_2 - l(a)_2 + l(a)_1 > u(a)_1 .$$

This contradicts the fact that a is bi-complete. We conclude that we have $u(a)_2 - l(a)_2 \leq u(a)_1 - l(a)_1$ and in the same way $u(b)_2 - l(b)_2 \leq u(b)_3 - l(b)_3$.

Given $x, y \in \mathcal{A}^*$, if there is a contact at slot 2 between the last two pieces of the heaps xab and yab (resp. xba and yba) then $\varphi(I\mathcal{M}(xa)) = \varphi(I\mathcal{M}(ya))$ (resp. $\varphi(I\mathcal{M}(xb)) = \varphi(I\mathcal{M}(yb))$). Given $x \in \mathcal{A}^*$, if there is a contact at slot 2 between the last two pieces of the heap xab (resp. xba) then it is also the case in the heap $xaab$ (resp. $xbba$). It implies that $\varphi(I\mathcal{M}(xaa)) = \varphi(I\mathcal{M}(xa))$ (resp. $\varphi(I\mathcal{M}(xbb)) = \varphi(I\mathcal{M}(xb))$). Let us set the ground shape to be $I = (-L, l(a)_2 - u(a)_2, -L)$ where the real $L > 0$ is assumed to be large enough to have $I\mathcal{M}(u) = (\mathbb{0}, l(a)_2 - u(a)_2, \mathbb{0})\mathcal{M}(u)$ for $u = a$ or b (see Figure 13 for an illustration). It implies that the slot 2 is a contact slot for ab and ba; hence we have $\varphi(I\mathcal{M}(aa)) = \varphi(I\mathcal{M}(a))$ and $\varphi(I\mathcal{M}(bb)) = \varphi(I\mathcal{M}(b))$. We deduce that

$$\varphi(\mathcal{H}) = \{\varphi(I), \varphi(I\mathcal{M}(a)), \varphi(I\mathcal{M}(b))\} \cup \{\varphi(I\mathcal{M}(abw)), \varphi(I\mathcal{M}(baw)), w \in \mathcal{A}^*\}$$

We assume for the moment that h_a/h_b is irrational. Let x be the jump word $(h_a, h_b, 0)$. Let us assume that the infinite heap abx has no gap on slots 1 and 3. Then, the heights of the pieces on slot 2 are:

- lower part of piece a: $\{nh_a - u(a)_2 + l(a)_2, n \in \mathbb{N}\}$;
- upper part of piece a: $\{nh_a, n \in \mathbb{N}\}$;
- lower part of piece b: $\{nh_b, n \in \mathbb{N}\}$;
- upper part of piece b: $\{nh_b + u(b)_2 - l(b)_2, n \in \mathbb{N}\}$.

Since h_a/h_b is irrational, by density of the points $\{nh_b \pmod{h_a}, n \in \mathbb{N}\}$ in the interval $[0, h_a]$, there exists a couple $(p, q) \in \mathbb{N}^2$ such that

$$ph_a - u(a)_2 + l(a)_2 < \ qh_b < \ ph_a \ .$$

This is a violation of the piling mechanism, see Figure 13-(i) for an illustration. Hence we conclude that there are some gaps on slot 1 or 3 in the heap abx. Let l_1 be such that there is no gap at slots 1 and 3 in the heap $abx[l_1 + 1]$ and there is a gap at slot 1 or 3 in the heap $abx[l_1 + 2]$. In Figure 13-(i), we have $l_1 = 3$ and $abx[l_1] = abbab$. Let l_2 be such that there is no gap at slots 1 and 3 in the heap $bax[l_2 + 1]$ and there is a gap at slot 1 or 3 in the heap $bax[l_2 + 2]$. Note that we have $l_1 \geq -1$ and $l_2 \geq -1$, and that it is possible to have $l_1 = -1$ and/or $l_2 = -1$.

Let us consider a heap abu (resp. bau), $u \in \mathcal{A}^*$. There are three possible cases.

(1) There is no gap at slots 1 and 3 in the heap and $u = x[n], n \leq l_1 + 1$ (resp. $u = x[n], n \leq l_2 + 1$). Let x_n be the n-th letter of x. If $u = x[l_1 + 1]$, then $\varphi(I\mathcal{M}(abx[l_1 + 1])) = \varphi(I\mathcal{M}(x_{l_1+1}))$ if $l_1 \geq 0$ and $\varphi(I\mathcal{M}(ab)) = \varphi(I\mathcal{M}(b))$ otherwise. Similarly we have $\varphi(I\mathcal{M}(bax[l_2 + 1])) = \varphi(I\mathcal{M}(x_{l_2+1}))$ if $l_2 \geq 0$ and $\varphi(I\mathcal{M}(ba)) = \varphi(I\mathcal{M}(a))$ otherwise.

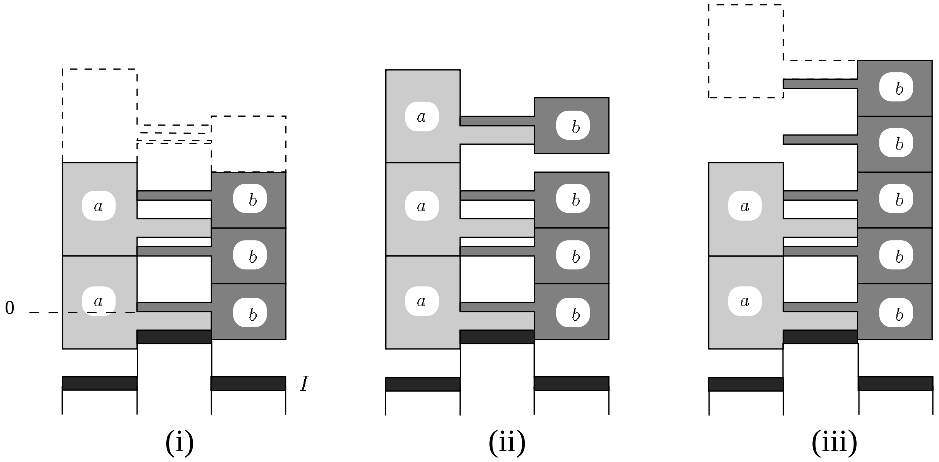

Fig. 13. (i) Heap $abx[3] = abbab$, (ii) heap $abbabab$, (iii) heap $abbabbb$.

(2) There is no gap at slots 1 and 3 and $u \neq x[|u|]$. In this case, we must have $u = x[n]a^m$ or $u = x[n]b^m$ with $m > 0, n \leq l_1 + 1$ (resp. $m > 0, n \leq l_2 + 1$). Assume we have $u = x[n]b^m$, the case $u = x[n]a^m$ being treated similarly. Since $u \neq x[|u|]$, in the heap $x[n]b^m a$, there is a contact at slot 2 between the last two pieces. We conclude that $\varphi(I\mathcal{M}(abx[n]b^m)) = \varphi(I\mathcal{M}(b))$ (resp. $\varphi(I\mathcal{M}(bax[n]b^m)) = \varphi(I\mathcal{M}(b))$). This case is illustrated in Figure 13-(iii) where $\varphi(I\mathcal{M}(abbabbb)) = \varphi(I\mathcal{M}(b))$.

(3) There is a gap somewhere in the heap at slot 1 or 3. This implies that we have in the heap u a contact at slot 2 between a piece a and a piece b, or between a piece b and a piece a. Considering the last couple (a, b) or (b, a) of pieces in contact at slot 2, we obtain (for abu, the case bau is treated similarly)

$$\varphi(I\mathcal{M}(abu)) = \varphi(I\mathcal{M}(abv)), \text{ or } \varphi(I\mathcal{M}(abu)) = \varphi(I\mathcal{M}(bav)),$$

where the heap abv, or bav, is such that there is no gap at slots 1 and 3. The heap abv, or bav, is in one of the two cases (1) or (2) above. Case (3) is illustrated in Figure 13-(ii) where $\varphi(I\mathcal{M}(abbabab)) = \varphi(I\mathcal{M}(ab))$, i.e. $u = babab$ and $v = e$.

To summarize, we have proved that

$$\begin{aligned}\varphi(\mathcal{H}) = \{&I, \varphi(I\mathcal{M}(a)), \varphi(I\mathcal{M}(b))\} \\ &\cup \{\varphi(I\mathcal{M}(abx[n])), 0 \leq n \leq l_1\} \cup \{\varphi(I\mathcal{M}(bax[n])), 0 \leq n \leq l_2\}\end{aligned}$$

The set $\varphi(\mathcal{H})$ is finite, hence we can apply the results of §3.2 to the contour-completed automaton.

Now, let us assume that h_a/h_b is rational. We still consider the jump word x with characteristics $(h_a, h_b, 0)$, which is now periodic, see §4. If the heap abx (or bax) has no gap on slots 1 and 3, then the schedule x is optimal (same argument as in §7.1). If the heaps abx and bax both have a gap somewhere on slot 1 or 3, the proof carries over exactly as in the case $h_a/h_b \notin \mathbb{Q}$.

The structure of the countour-completed automaton can be deduced from the above proof. For simplicity, we denote the state $\varphi(I\mathcal{M}(w))$ by w, and we use the convention $a' = b, b' = a$. For $0 \le n \le l_1 - 1$, there is a transition $abx[n] \xrightarrow{x_{n+1}} abx[n+1]$ and a transition $abx[n] \xrightarrow{x'_{n+1}} x'_{n+1}$. For $0 \le n \le l_2 - 1$, there is a transition $bax[n] \xrightarrow{x_{n+1}} bax[n+1]$ and a transition $bax[n] \xrightarrow{x'_{n+1}} x'_{n+1}$. In

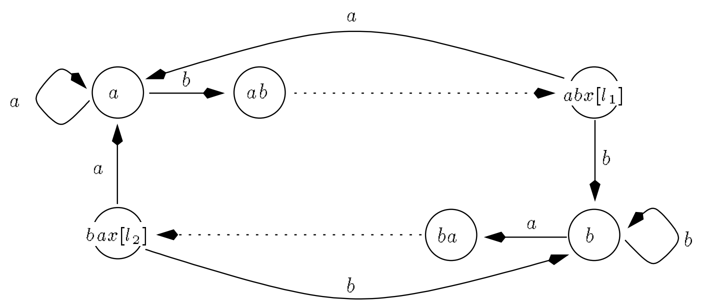

Fig. 14. Outline of the contour-completed automaton.

Figure 14, we have represented an outline of the contour-completed automaton in the case $l_1 > 0, l_2 > 0$ (ingoing and outgoing arrows as well as some arcs are missing, and the multiplicities have been omitted).

Using the above analysis, we can get the value of the multiplicities in the contour-completed automaton. Doing this, we obtain that there is a circuit of minimal mean weight in the contour-completed automaton of label either: $a, b, abx[l_1], bax[l_2]$, or $abx[l_1]bax[l_2]$, with the conventions $x[0] = e, abx[-1] = a, bax[-1] = b$. Hence, one of the following schedules is optimal: $\{a^\omega, b^\omega, (abx[l_1])^\omega, (bax[l_2])^\omega, (abx[l_1]bax[l_2])^\omega\}$. It remains to be proved that $(abx[l_1])^\omega, (bax[l_2])^\omega$, and $(abx[l_1]bax[l_2])^\omega\}$ are balanced.

We are going to prove that $(bax[l_2]abx[l_1])^\omega$ is balanced. We treat the case $l_1 \ge 0$ and $l_2 \ge 0$. If we have l_1 or l_2 equal to -1, the argument can be easily adapted. Due to the definition of l_1, the following intervals are all disjoint (visually, they correspond to the portions of the second column occupied by the pieces in the heap $abx[l_1]$. We consider open intervals in $\mathcal{I}_a$ and closed ones in $\mathcal{I}_b$ in order to ensure that the first interval in $\mathcal{I}_a$ and $\mathcal{I}_b$ are indeed disjoints):

$$\begin{aligned}\mathcal{I}_a &= \{(nh_a + l(a)_2 - u(a)_2, nh_a),\ 0 \le n \le |x[l_1]|_a\},\\ \mathcal{I}_b &= \{[nh_b, nh_b + u(b)_2 - l(b)_2],\ 0 \le n \le |x[l_1]|_b\}\,.\end{aligned}$$

In the same way, the following intervals are all disjoint (up to the minus sign, they correspond to the portions of the second column occupied by the pieces in the heap $bax[l_2]$):

$$\mathcal{I}'_a = \{(-nh_a + l(a)_2 - u(a)_2, -nh_a),\ 0 \le n \le |x[l_2]|_a\},$$
$$\mathcal{I}'_b = \{[-nh_b, -nh_b + u(b)_2 - l(b)_2],\ 0 \le n \le |x[l_2]|_b\}\,.$$

An illustration of the intervals in $\mathcal{I}_a, \mathcal{I}_b, \mathcal{I}'_a$ and $\mathcal{I}'_b$ is provided in Figure 15-(i)-(ii). Let us label the intervals in $\mathcal{I}_a \cup \mathcal{I}'_a$ by a and the ones in $\mathcal{I}_b \cup \mathcal{I}'_b$ by b. If we read the sequence of labels from bottom to top, we obtain the word $\tilde{x}[l_2]abx[l_1]$, where $\tilde{x}[l_2]$ is the mirror word of $x[l_2]$ (the *mirror word* of the word $u = u_1u_2\cdots u_n$ is the word $\tilde{u} = u_nu_{n-1}\cdots u_1$). Setting $n_a = |x[l_1]|_a + 1, n_b = |x[l_1]|_b + 1, n'_a = |x[l_2]|_a + 1$, and $n'_b = |x[l_2]|_b + 1$, we have (by definition of l_1 and l_2)

$$(n_ah_a + l(a)_2 - u(a)_2, n_ah_a) \cap [n_bh_b, n_bh_b + u(b)_2 - l(b)_2] \neq \emptyset,$$
$$(-n'_ah_a + l(a)_2 - u(a)_2, -n'_ah_a) \cap [-n'_bh_b, -n'_bh_b + u(b)_2 - l(b)_2] \neq \emptyset\,.$$

Let us choose $t \in (-n'_ah_a + l(a)_2 - u(a)_2, -n'_ah_a) \cap [-n'_bh_b, -n'_bh_b + u(b)_2 - l(b)_2]$. Let us consider the set

$$\mathcal{S} = \{t + nh_a, 1 \le n \le n_a + n'_a - 1\} \cup \{t + nh_b, 1 \le n \le n_b + n'_b - 1\}\,.$$

By construction, each real of $\mathcal{S}$ is in a different interval of $\mathcal{I}_a, \mathcal{I}_b, \mathcal{I}'_a$ or $\mathcal{I}'_b$. Hence, if we read the sequence of labels associated with $\mathcal{S}$ from bottom to top, we obtain $x[l_1 + l_2 + 2]$. Also by construction, we have $t + (n_a + n'_a)h_a \in (n_ah_a + l(a)_2 - u(a)_2, n_ah_a)$ and $t + (n_b + n'_b)h_b \in [n_bh_b, n_bh_b + u(b)_2 - l(b)_2]$.

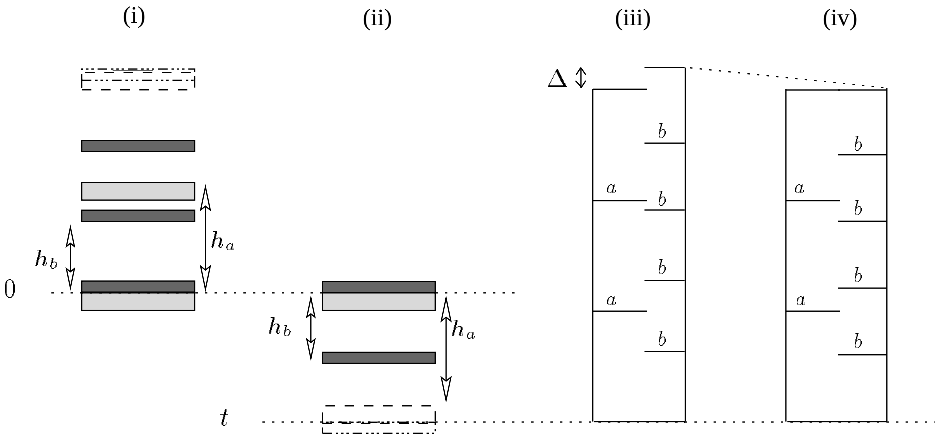

Fig. 15. Illustration of the proof; here $x[l_1] = bab$ and $x[l_2] = b$.

Let us set $m_a = n_a + n'_a$ and $m_b = n_b + n'_b$. We define $\Delta = m_ah_a - m_bh_b$, see Figure 15-(iii). By construction, we have either $(*)$ or $(**)$:

$(*)$ If $\Delta > 0$ then $\Delta < nh_a - n'h_b$ for each n, n' such that $1 \le n < m_a, 1 \le n' < m_b, nh_a - n'h_b > 0$.

$(**)$ If $\Delta < 0$ then $\Delta > nh_a - n'h_b$ for each n, n' such that $1 \le n < m_a, 1 \le n' < m_b, nh_a - n'h_b < 0$.

Let us assume that $\Delta < 0$ (the case of Figure 15-(iii)). The other case is treated similarly. Because of the property $(**)$, the sequence of a's and b's corresponding to $\mathcal{S}$ is the same as the one corresponding to

$$\{t + nh_a, 1 \leq n \leq m_a - 1\} \cup \{t + nh_b(m_a h_a)/(m_b h_b), 1 \leq n \leq m_b - 1\} .$$

Equivalently the jump words $(h_a, h_b, 0)$ and $(h_a, h_b(m_a h_a)/(m_b h_b), 0) = (h_a, h_a m_a/m_b, 0)$ have the same prefix of length $l_1 + l_2 + 2$. If we decide to read double points as ba (see §4), then the jump word z with characteristics $(h_a, h_a m_a/m_b, 0)$ is a balanced and periodic word, which is equal to $(\tilde{x}[l_2]abx[l_1]ba)^\omega$. A palindrome is a word equal to its mirror word. The above construction shows that $\tilde{x}[l_2]abx[l_1]$ is a palindrome (for instance, in Figure 15-(iv), the sequences of a's and b's read from bottom to top, and top to bottom, are the same). It implies that it is impossible to have $l_1 = l_2$ (since $\tilde{x}[l]abx[l]$ is never a palindrome). By the same type of arguments, we can prove that $x[l_1]$ and $x[l_2]$ are also palindromes. Hence we have $x[l_2]abx[l_1]ba = \tilde{x}[l_2]abx[l_1]ba$ and we conclude that $(x[l_2]abx[l_1]ba)^\omega$ is balanced.

The fact that $(abx[l_1])^\omega$ and $(bax[l_2])^\omega$ are balanced is proved in a similar way.

7.5 Greedy scheduling

We treat completely an instance of the jobshop described in the introduction, see Figure 1. The durations of the activities are assumed to be $\alpha_1 = \alpha^4 (= 4 \times \alpha)$, $\alpha_2 = \alpha$, $\beta_1 = \alpha$ and $\beta_2 = 1 - \alpha$. We assume that $1/15 < \alpha < 1/11$. The model corresponds to case (ii) in §7.4 above.

The contour-completed automaton of $\mathcal{H} = (\pi(I\mathcal{M}(a)), \mathcal{M}, \mathbb{1})$ is represented in Figure 16. The labels of the simple circuits are a, b, ba, ba^2 and ba^3. Their

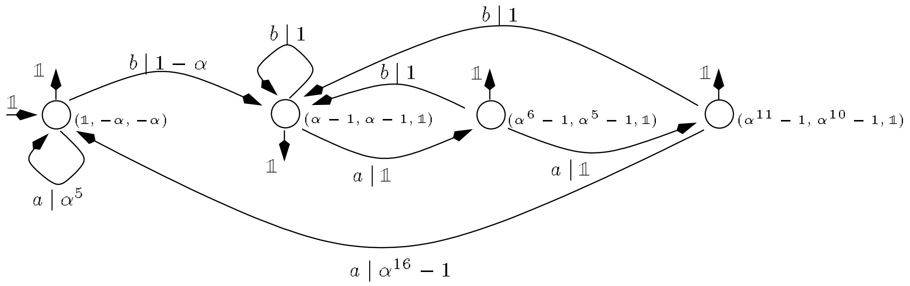

Fig. 16. Contour-completed automaton.

respective mean weights are $\alpha^5, 1, 1/2, 1/3$ and $\alpha^{15/4}$. Hence the label of the circuit of minimal mean weight is ba^2 if $4/45 \leq \alpha$ and ba^3 if $\alpha \leq 4/45$. We conclude that an optimal schedule is

$$(ba^3)^\omega \text{ if } 1/15 < \alpha \leq 4/45, \quad (ba^2)^\omega \text{ if } 4/45 \leq \alpha < 1/11 .$$

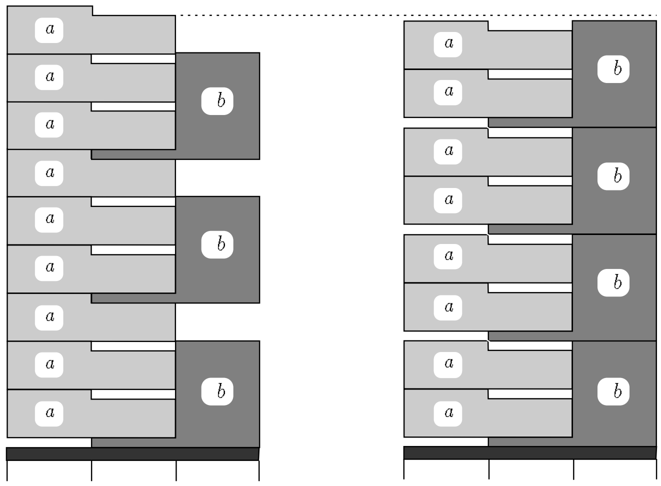

Fig. 17. Model with $\alpha = 8/89$, greedy schedule and optimal schedule.

The *greedy scheduling* consists in always allocating the resource to the first task which is ready to use it (i.e. $w[n+1] = w[n]a$, resp. $w[n+1] = w[n]b$, if we have $x_{\mathcal{H}}(w[n])_1 < x_{\mathcal{H}}(w[n])_3$, resp. $x_{\mathcal{H}}(w[n])_3 < x_{\mathcal{H}}(w[n])_1$). Here the greedy schedule is always $(ba^3)^\omega$. We conclude that greedy scheduling is suboptimal in the case $\alpha \in (4/45, 1/11)$, see Figure 17.

This is in sharp contrast with a result from [23] §IV. There, the optimal problem is studied for the model of Figure 1, but the authors consider a slightly different criterion: minimization of the idle time of the resource. They show that greedy schedules are indeed optimal for this criterion.

7.6 Ratio constraints

In [25,22,23], the authors were primarily interested in the following *constrained optimal problem*: *Find* $w \in \mathcal{A}^\omega$ *minimizing* $\lim_n y_{\mathcal{H}}(w[n])/n$ *while satisfying* $\lim_n |w[n]|_a/n = \gamma$ *where* $\gamma \in [0,1]$ *is some given ratio constraint.*

In a manufacturing model, the motivation is to maximize the throughput while meeting a given production ratio. For this constrained problem, and for the model of Figure 1, it is proved in [22,23] that the optimal schedule is always the jump word $(1-\gamma, \gamma, 0)$. Two points are worth being noticed. First, the optimal schedule is balanced and when $\gamma \in \mathbb{Q}$, it is of the form u^ω where u is the shortest balanced word meeting the ratio constraint. Second, the optimal schedule does not depend on the timings of the model ($\alpha_1, \alpha_2, \beta_1$ and β_2 in Figure 1).

These two properties depend heavily on the specific shape of the pieces in the model of Figure 1. They are not satisfied in a general heap model with two pieces, as shown below.

Example 17 *Consider the model of Example 15. We look at the constrained optimal problem with ratio* $1/2$. *The optimal schedule of length* $2n, n \in \mathbb{N}^*$,

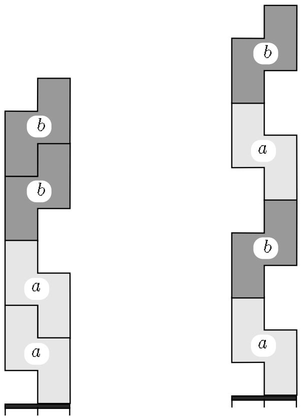

Fig. 18. Optimal and worst schedule of ratio 1/2 and length 4.

is $a^n b^n$ *(or* $b^n a^n$*) as illustrated on Figure 18. A possible optimal schedule is* $aba^2b^2 \cdots a^n b^n \cdots$. *No infinite balanced word with ratio* $1/2$ *is optimal. Here, the schedule* $(ab)^\omega$, *whose period is the shortest balanced word meeting the constraint, is not an optimal but a worst case schedule! Examples in the same spirit appear in [14], § VI-1 and in [20], §5.1.*

8 Heap Models with Two Pieces: Average Case

In this section, products have to be interpreted in the field $(\mathbb{R}, +, \times)$. We still assume that $l(a) \neq u(a)$ and $l(b) \neq u(b)$, otherwise the average problem becomes trivial.

As in §7, the distinction between 'determinizable' and 'non-determinizable' automata is important. For the 'determinizable' case, it is easy to check that the automata obtained in §7.1-7.4 are all irreducible. Hence we obtain ρ_E by applying Prop. 6. Below, we illustrate this case on one example. There are two cases where the heap automaton is 'non-determinizable', see §7. In one case, we come up with an explicit formula for ρ_E and in the other case, we express it as an infinite series.

Determinizable automaton. We consider the heap automaton $\mathcal{H}$ of §7.5. Let $\{p(a), p(b)\}$ be the probability distribution of the pieces. The contour-completed automaton is represented in Figure 16. The corresponding transition matrix is (see Prop. 6):

$$P = \begin{pmatrix} p(a) & 1-p(a) & 0 & 0 \\ 0 & 1-p(a) & p(a) & 0 \\ 0 & 1-p(a) & 0 & p(a) \\ p(a) & 1-p(a) & 0 & 0 \end{pmatrix} .$$

Its stationary distribution is $\pi = (p(a)^3, 1-p(a), p(a)-p(a)^2, p(a)^2-p(a)^3)$. We conclude that we have, Prop. 6,

$$\rho_E(\mathcal{H}) = (-10\alpha+1)p(a)^4 + (15\alpha-1)p(a)^3 - p(a) + 1 .$$

This formula is valid for $1/15 < \alpha < 1/11$, see §7.5. For instance, in the case $\alpha = 8/89$ (the one of Figure 17), we have

$$\rho_E(\mathcal{H}) = \frac{9}{89}p(a)^4 + \frac{31}{89}p(a)^3 - p(a) + 1, \ \ \min_{p(a)} \rho_E(\mathcal{H}) = 0.417 \ \text{ for } \ p(a) = 0.849.$$

Case $R(a) = \{1\}, R(b) = \{2\}$. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $x_n, n \in \mathbb{N}^*$, be independent random variables such that $P\{x_n = a\} = p(a)$ and $P\{x_n = b\} = p(b)$. We set $x_{\mathcal{H}}(n) = I\mathcal{M}(x_1 \cdots x_n)$. The processes $x_{\mathcal{H}}(n)_1$ and $x_{\mathcal{H}}(n)_2$ are transient random walks with respective drifts $p(a)h_a$ and $p(b)h_b$. We deduce immediately that $\rho_E(\mathcal{H}) = \max(p(a)h_a, p(b)h_b)$.

Case $R(a) = \{1,2\}, R(b) = \{2,3\}$. We consider the case $R(a) = \{1,2\}, R(b) = \{2,3\}, u(a)_2 = l(a)_2, u(b)_2 = l(b)_2$ and $h_a/h_b \notin \mathbb{Q}$. A simple but lengthy computation provides the following formula (the details are available from the authors on request). Let us denote by u $(= u_1u_2\cdots)$ the jump word $(h_a, h_b, 0)$. We use the convention $a' = b, b' = a$. We set $\delta(a) = h_a/(h_a+h_b)$, $\delta(b) = h_b/(h_a+h_b)$ and $c_n = p(a)^{\lfloor n\delta(b)\rfloor}p(b)^{\lfloor n\delta(a)\rfloor}/p(u_n)$. We have

$$\rho_E(\mathcal{H}) = (h_a+h_b)\frac{\sum_{n=1}^{\infty} c_n p(u_n')\delta(u_n')\left(p(u_n)\lfloor\delta(u_n)n\rfloor + 1\right)}{\sum_{n=1}^{\infty} c_n} . \tag{33}$$

One can obtain approximations of $\rho_E(\mathcal{H})$ by truncating the infinite sums. Computations of ρ_E for closely related models are carried out in [26].

9 Conclusion: Heap Models with Three or More Pieces

As recalled in the introduction, the optimal problem for a heap model with an arbitrary number of rational pieces ($\forall a \in \mathcal{A}, u(a), l(a) \in \mathbb{Q}_{\max}^{\mathcal{R}}$) is solved in [21]. In Theorem 14, the case of a heap model with two general pieces is treated. We recall the results in the table below.

	$\|\mathcal{A}\| = 2$	$\|\mathcal{A}\| > 2$
$\mathbb{Q}_{\max}^{\mathcal{R}}$	periodic	periodic
$\mathbb{R}_{\max}^{\mathcal{R}}$	periodic or Sturmian	?

Characterizing optimal schedules is an open problem for models with three pieces or more. Generalized versions of jump words appear naturally in some models. Let $\mathcal{A} = \{a_1, a_2, \cdots, a_k\}$ be the alphabet. We consider $\alpha_i \in \mathbb{R}_+^*, \gamma_i \in \mathbb{R}_+, \gamma_i < \alpha_i$, for $i \in \{1, \ldots, k\}$. We label the points $\{n\alpha_i + \gamma_i, n \in \mathbb{N}^*\}$ by a_i and we consider the set $\cup_{i=1}^k \{n\alpha_i + \gamma_i, n \in \mathbb{N}^*\}$ in its natural order. The infinite sequence of labels is called the *(hypercubic) billiard sequence* with characteristics $(\alpha_i, \gamma_i, i = 1, \ldots, k)$, see [3,6]. Now let us consider the heap model $\mathcal{H} = (\mathcal{A}, \{1, \ldots, k+1\}, R, u, l, \mathbb{1})$ with $R(a_i) = \{i, k+1\}, u(a_i)_i - l(a_i)_i = h_i > 0$ and $u(a_i)_{k+1} = l(a_i)_{k+1}$. Using an argument similar to the one in §7.4, we obtain that the billiard sequence with characteristics $(h_i, 0, i = 1, \ldots, k)$ is an optimal schedule. A similar result is obtained for the heap model $(\mathcal{A}, \{1, \ldots, k\}, R, u, l, \mathbb{1})$ with $R(a_i) = i$.

Further research. During the reviewing process of this work, alternative proofs of Theorem 14 as well as further developments have been proposed in [30,11]. The methods in [11] also enable to refute the Lagarias-Wang finiteness conjecture [27].

Acknowledgements

The authors thank Thierry Bousch, Stéphane Gaubert, Bruno Gaujal and Colin Sparrow for stimulating discussions on the subject. The detailed comments of an anonymous referee have also helped in improving the paper.

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