Tilings and rotations on the torus: a two dimensional generalization of Sturmian sequences

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Abstract: *We study a two dimensional generalization of Sturmian sequences corresponding to an approximation of a plane: these sequences are defined on a three-letter alphabet and code a two dimensional tiling obtained by projecting a discrete plane. We show that these sequences code a $\mathbb{Z}^2$-action generated by two rotations on the unit circle. We first deduce a new way of computing the rectangle complexity function. Then we provide an upper bound on the number of frequencies of rectangular factors of given size.*

Keywords: discrete plane, generalized Sturmian sequences, symbolic dynamics, combinatorics on words, $\mathbb{Z}^2$-action.

1 Introduction

1.1 Discrete planes and tilings

Discrete planes and tilings arising from their projection, appear in numerous fields. In particular, such tilings are included in the class of tilings obtained by the "cut and project" method, like the Penrose tiling [13], see also [30, 32]. These tilings have deep connections with quasicrystals (see for instance [12, 36]).

Discrete planes and lines are also studied in digital geometry and correspond, in the rational case, to the notion of an arithmetic plane (see for instance [31] for an arithmetic and algorithmic study or [20], for a topological approach).

The recognition of discrete lines and planes is a classical problem in computer imagery. Basic questions are to obtain a characterization of discrete planes and to recognize whether a set of points is contained in a discrete plane. These two questions have mathematical answers, but the problem for computer scientists is to find fast algorithms to check these properties. Our work might be applied in this field, in particular, with regard to the introduced property of balance and to the minimality of the complexity (see also [43]).

Discrete planes correspond to the stepped surface introduced and studied by Ito and Ohtsuki in [25, 26] from a number-theoretic point of view: more

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precisely, Ito and Ohtsuki study a generating method of a stepped surface by using the multi-dimensional continued fraction algorithm of Jacobi-Perron and by introducing substitutions on square faces.

This article studies a coding by a two dimensional sequence of an aperiodic tiling of the plane. Consider the set of all unit cubes, with vertices at integer lattice points, which intersect a given plane. The *discrete plane* approximating this plane is the (upper or lower) surface of the union of these unit cubes. The discrete plane thus consists of three kinds of square faces, orientated according to the three coordinate planes. After projection, we obtain a tiling of the plane by three kinds of diamonds, being the projections of the square faces. We prove that this tiling is associated with a $\mathbb{Z}^2$-lattice. We code this tiling over a two dimensional sequence defined on a three-letter alphabet. Such a sequence has been introduced by the second author in [42]. The purpose of this paper is to show that this two dimensional sequence is a coding of a $\mathbb{Z}^2$-action given by two rotations on the unit circle $\mathbb{R}/\mathbb{Z}$ and then to deduce metric and topological properties.

This article is organized as follows. We first recall basic results concerning Sturmian sequences. The second section contains a brief summary of the discrete line construction. We introduce in the third section a two dimensional generalization of Sturmian sequences. We show that such a sequence codes a $\mathbb{Z}^2$-action generated by two rotations on the unit circle and we introduce a property of balance. In Section 4, we characterize the rectangular factors of given size in terms of a partition of the unit circle. In Section 5.2, we provide a new way to compute the rectangle complexity and we show that the complexity function satisfies $P(m,n) = mn + m + n$, for all (m,n) (this result was already given in [42], see also [31] for the rational case). By using a two dimensional version of the three distance theorem (proved by Geelen and Simpson in [21]), we also deduce, that the frequencies of rectangular factors of size (m,n) take at most $\min(m,n) + 5$ values (Section 5.2). We also prove some properties of uniform recurrence on the language of rectangular factors (Section 5.3). In the last section, we explore the notion of minimal complexity for two dimensional sequences. In particular, we provide an example of a two dimensional sequence of complexity $P(m,n) = mn + n$, for every (m,n), which is uniformly recurrent and which has no rational periodic direction.

1.2 Sturmian sequences

A *Sturmian sequence* is defined as a unidirectional sequence of complexity $p(n) = n + 1$, for every n. Recall that the *complexity function* counts the number of distinct factors of given length. Note that a sequence whose complexity satisfies $p(n) \leq n$, for some n, is ultimately periodic (see [28]). Sturmian sequences thus have minimal complexity among non-ultimately periodic sequences. Sturmian sequences are also characterized by the following geometric and combinatorial properties. A recent account on the subject can be found in [11, 27].

- A Sturmian sequence is the coding of the positive orbit of a point x of the unit circle under an irrational rotation of angle α, say, with respect to a partition into two semi-intervals of size α and $1-\alpha$ (see [29]).

- Sturmian sequences are codings of trajectories of irrational slope in a square billiard table obtained by coding horizontal sides by the letter 0 and vertical sides by the letter 1.

- Sturmian sequences are defined by approximation of a line of irrational slope, i.e., by coding a discrete line (see Section 2).

- Sturmian sequences are exactly the non-ultimately periodic balanced sequences over a two-letter alphabet. A *balanced sequence* is such that the absolute values of the differences between the number of occurrences of a letter in any two factors of the same length are at most 1.

By abuse of notation, we call in the sequel a two-sided sequence Sturmian if the restriction of any translate over $\mathbb{N}$ is Sturmian. This means that it is the coding of the orbit (in both directions) of a point x of the unit circle under an irrational rotation of angle α, say, with respect to a partition into two half-open intervals of size α and $1-\alpha$. Such a sequence has complexity $p(n+1) = n+1$, for every n, but the converse is false, as illustrated by the example of the sequence $\ldots 0 \ldots 010 \ldots 0 \ldots$

Unidimensional generalizations with values in a three-letter alphabet have been given by:

a) playing billiards in a cube, one thus obtains the complexity function $p(n) = n^2 + n + 1$ (see [3] and, for the n-dimensional case, see [5]),

b) by considering sequences of complexity $p(n) = 2n+1$ with an extra combinatorial condition [4],

c) by introducing the notion of episturmian words, which generalizes palindromic properties of Sturmian sequences [17],

d) or by considering well-balanced sequences [22, 24].

Let us see in what respect the two dimensional sequences studied here provide a generalization of Sturmian sequences. We extend the construction with discrete lines to a higher-dimensional space (Sections 2 and 3). We show furthermore that these sequences code a $\mathbb{Z}^2$-action generated by two rotations on the unit circle. These sequences also satisfy some property of balance, that we describe in Section 3.2. Finally we introduce in Section 6 sequences defined on a two-letter alphabet, and produced as a letter-to-letter projection of the two dimensional Sturmian sequences. We conjecture these sequences to be of “minimal” rectangular complexity among two dimensional sequences, which are uniformly recurrent but not periodic. Furthermore, they can be described among uniformly recurrent sequences as those sequences of complexity $mn+n$ [7]. They also show some interesting properties of palindromy generalizing the Sturmian case [8].

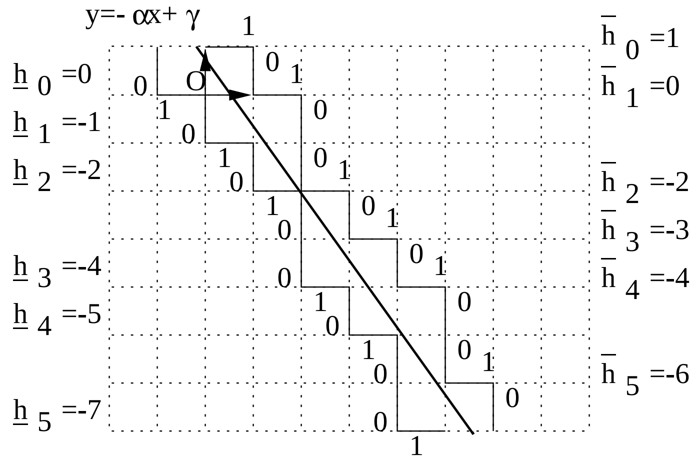


Figure 1: Discrete line

2 Discrete line

Let us first recall the classical connection between approximations of a line in the plane and Sturmian sequences (see for instance [10] or [27]). Let $\mathcal{D}$: $y = -\alpha x + \gamma$ be a line in $\mathbb{R}^2$. We can assume $\alpha > 1$ without loss of generality, by permuting the coordinate axes if necessary and $0 \le \gamma < 1$, by translating the axes by integers (we will come back to these restrictions in the Remark at the end of this section). We associate to $\mathcal{D}$ two discrete lines, namely an upper discrete line $\overline{D}$ and a lower discrete line $\underline{D}$, by approximating $\mathcal{D}$ by vertical and horizontal edges of length 1 as follows (see Figure 1).

Definition 1 *Let $\overline{\mathcal{S}}$ (respectively $\underline{\mathcal{S}}$) be the set of translates of the fundamental square with integer vertices that intersect the lower half-plane $y < -\alpha x + \gamma$ (respectively the upper half-plane $y > -\alpha x + \gamma$).The* discrete line *$\overline{D}$ (respectively $\underline{D}$) is defined as the topological boundary of $\overline{\mathcal{S}}$ (respectively $\underline{\mathcal{S}}$).*

Define $\overline{h}_n = \lceil -\alpha n + \gamma \rceil$ and $\underline{h}_n = \lfloor -\alpha n + \gamma \rfloor$, for all $n \in \mathbb{Z}$. The integer $\overline{h}_n$ (respectively $\underline{h}_n$) gives the height of the n^{th} horizontal edge in the discrete line $\overline{D}$ (respectively $\underline{D}$).

Let us associate with the discrete lines $\overline{D}$ and $\underline{D}$, respectively, the two two-sided sequences $\overline{u} = (\overline{u}_n)_{n\in\mathbb{Z}}$ and $\underline{u} = (\underline{u}_n)_{n\in\mathbb{Z}}$, with values in $\{0,1\}$, defined by

$$\overline{u}_n = 1 \textit{ if and only if there exists } j \in \mathbb{Z} \textit{ such that } n = \overline{h}_0 - \overline{h}_j + j,$$

$$\underline{u}_n = 1 \textit{ if and only if there exists } j \in \mathbb{Z} \textit{ such that } n = \underline{h}_0 - \underline{h}_j + j.$$

The sequences $\overline{u}$ and $\underline{u}$ are respectively called the upper and lower coding *of the line $\mathcal{D}$.*

Note that $0 \le \gamma < 1$ implies that $\overline{h}_0 = 0$ or 1 and that $\underline{h}_0 = 0$. Furthermore $\overline{u}_0 = \underline{u}_0 = 1$, by definition.

As $\alpha > 1$, then $-\overline{h}_{j+1} + j + 1 > -\overline{h}_j + j$, for any $j \in \mathbb{Z}$. The same holds for the integers $\underline{h}_j$. Hence, for any integer n, there exists a unique $\overline{j} \in \mathbb{Z}$ such that $\overline{h}_0 - \overline{h}_{\overline{j}} + \overline{j} \le n < \overline{h}_0 - \overline{h}_{\overline{j}+1} + \overline{j} + 1$. We define $\underline{j}$ in the same way.

Roughly speaking the two sequences $\overline{u}$ and $\underline{u}$ code respectively the discrete lines $\overline{\mathcal{D}}$ and $\underline{\mathcal{D}}$, by 1 for an horizontal edge and by 0 for a vertical edge (see Figure 1): actually, if $\overline{u}_n = 1$, then $\overline{u}_n$ codes the horizontal edge of endpoints $(\overline{j}, \overline{h}_{\overline{j}})$ and $(\overline{j}+1, \overline{h}_{\overline{j}})$ of the discrete line, and if $\overline{u}_n = 0$, then $\overline{u}_n$ codes the vertical edge of endpoints $(\overline{j}+1, -n+\overline{j}+\overline{h}_0+1)$ and $(\overline{j}+1, -n+\overline{j}+\overline{h}_0)$; a similar description holds for the sequence $\underline{u}$.

The following proposition expresses the two sequences $\overline{u}$ and $\underline{u}$ as codings of rotations of the unit circle.

In all that follows R_α denotes the rotation of angle α defined on the unit circle by:

$$R_\alpha(x) = x + \alpha \text{ (modulo 1)}.$$

Except where stated otherwise, every quantity is considered modulo 1.

Proposition 1 *Let $\overline{u}$ and $\underline{u}$ be respectively the upper and the lower coding of the line $\mathcal{D} : y = -\alpha x + \gamma$, where $\alpha > 1$ and $0 \le \gamma < 1$. Let $\overline{\gamma} = \gamma$, if $\gamma \neq 0$ and $\overline{\gamma} = 1$, otherwise.*

The sequences $\overline{u}$ and $\underline{u}$ satisfy

$$\overline{u}_n = i \Longleftrightarrow R^n_{\frac{1}{1+\alpha}}\left(\frac{\overline{\gamma}}{1+\alpha}\right) \in \overline{I}_i,$$

$$\underline{u}_n = i \Longleftrightarrow R^n_{\frac{1}{1+\alpha}}\left(\frac{\gamma}{1+\alpha}\right) \in \underline{I}_i,$$

where

$$\overline{I}_1 = \left]0, \frac{1}{1+\alpha}\right] \text{ and } \overline{I}_0 = \left]\frac{1}{1+\alpha}, 1\right],$$

$$\underline{I}_1 = \left[0, \frac{1}{1+\alpha}\right[\text{ and } \underline{I}_0 = \left[\frac{1}{1+\alpha}, 1\right[.$$

Proof Consider the sequence $\overline{u}$. Suppose that $\overline{u}_n = 1$. There exists a unique $\overline{j} \in \mathbb{Z}$ such that

$$n = h_0 - \overline{h}_{\overline{j}} + \overline{j},$$

i.e.,

$$n = \overline{j} + \lceil \gamma \rceil - \lceil -\alpha\overline{j} + \gamma \rceil.$$

Note that $\overline{\gamma} = \gamma - \lceil \gamma \rceil + 1$. We thus have

$$(1+\alpha)\overline{j} < n + \overline{\gamma} \le (1+\alpha)\overline{j} + 1,$$

i.e.,

$$\overline{j} < \frac{n+\overline{\gamma}}{1+\alpha} \le \overline{j} + \frac{1}{1+\alpha},$$

which implies

$$\frac{n+\overline{\gamma}}{1+\alpha} \in \left]0, \frac{1}{1+\alpha}\right] \quad \text{modulo } 1.$$

Suppose now that $\overline{u}_n = 0$. There exists a unique $\overline{j} \in \mathbb{Z}$ such that

$$\overline{h}_0 - \overline{h}_{\overline{j}} + \overline{j} + 1 \leq n \leq \overline{h}_0 - \overline{h}_{\overline{j}+1} + \overline{j},$$

which implies similarly that

$$\frac{n}{1+\alpha} + \frac{\overline{\gamma}}{1+\alpha} \in \left]\frac{1}{1+\alpha}, 1\right] \quad \text{modulo } 1.$$

The same reasoning applies to $\underline{u}$. ■

Remarks

- This construction is to be compared with the notions of *mechanical sequences* and *cutting sequences* (see for instance [27]).
- When α is a rational number, both sequences $(\overline{u}_n)_{n\in\mathbb{Z}}$ and $(\underline{u}_n)_{n\in\mathbb{Z}}$ are periodic.
- When α is an irrational number, then both sequences $(\overline{u}_n)_{n\in\mathbb{N}}$ and $(\underline{u}_n)_{n\in\mathbb{N}}$ are easily seen to be Sturmian sequences. Actually a Sturmian sequence is the coding of a point of the unit circle under the action of an irrational rotation of angle α, say, with respect either to the partition $\{[0, 1-\alpha[, [1-\alpha, 1[\}$ or to the partition $\{]0, 1-\alpha],]1-\alpha, 1]\}$. Suppose that the letter 1 codes the interval of length α. We thus have a bijection between the set of Sturmian sequences (defined on $\mathbb{N}$) that begin with the letter 1 and the set of the restrictions over $\mathbb{N}$ of the upper and lower codings of the lines $\mathcal{D} : y = -\alpha x + \gamma$, with α irrational number, $\alpha > 1$ and $0 \leq \gamma < 1$.
- We deduce from the classical properties of Sturmian sequences that the two sequences $\overline{u}$ and $\underline{u}$ have the same set of factors. More generally the set of factors of a coding of a discrete line only depends on the slope α of the discrete line (see for instance [27]).
- With the previous notations, we can deduce from the proof of Proposition 2.1 the expression for $\overline{j}$ and $\underline{j}$:

  $$\overline{j} = \left\lceil \frac{n+\overline{\gamma}}{1+\alpha} \right\rceil - 1,$$

  $$\underline{j} = \left\lfloor \frac{n+\gamma}{1+\alpha} \right\rfloor.$$

  Furthermore, given any irrational number α and any γ, there exists at most one n for which $\overline{j} \neq \underline{j}$: such an integer n exists if and only if $\gamma \in (1+\alpha)\mathbb{Z} + \mathbb{Z}$; we thus have if $\gamma \neq 0$, $\underline{j} = \frac{n+\gamma}{1+\alpha}$, $\overline{u}_n = 0$, $\overline{u}_{n+1} = 1$, $\underline{u}_n = 1$, $\underline{u}_{n+1} = 0$, and $\overline{u}_k = \underline{u}_k$, for any $k \neq n, n+1$. If $\gamma = 0$, then $\overline{u}_0 = 1$, $\overline{u}_{-1} = 0$, $\underline{u}_0 = 1$, $\underline{u}_1 = 0$, and $\overline{u}_{k-1} = \underline{u}_k$, for any $k \neq 0, 1$. If $\gamma \notin (1+\alpha)\mathbb{Z} + \mathbb{Z}$, then both sequences are equal.

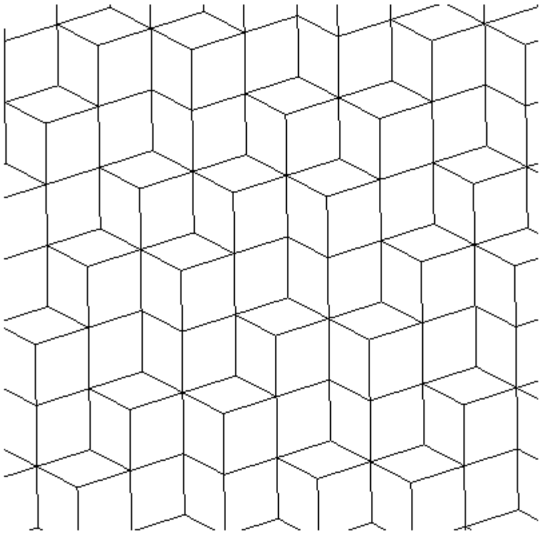


Figure 2: A discrete plane

3 Discrete plane

3.1 Construction

Let us generalize this construction to approximations in $\mathbb{R}^3$. Let $\mathcal{P} : z = -\alpha x - \beta y + \gamma$ be a plane in $\mathbb{R}^3$ endowed with the canonical basis $(\vec{e_1}, \vec{e_2}, \vec{e_3})$. We can still assume $\alpha > 1$, $\beta > 1$, and $0 \leq \gamma < 1$, without loss of generality. We associate to the plane $\mathcal{P}$ two discrete planes, namely an upper discrete plane $\overline{P}$ and a lower discrete plane $\underline{P}$, by approximating $\mathcal{P}$ by unit square faces as follows (see Figure 2 and 3.a). This construction corresponds to the stepped surface introduced by Ito and Ohtsuki in [25, 26].

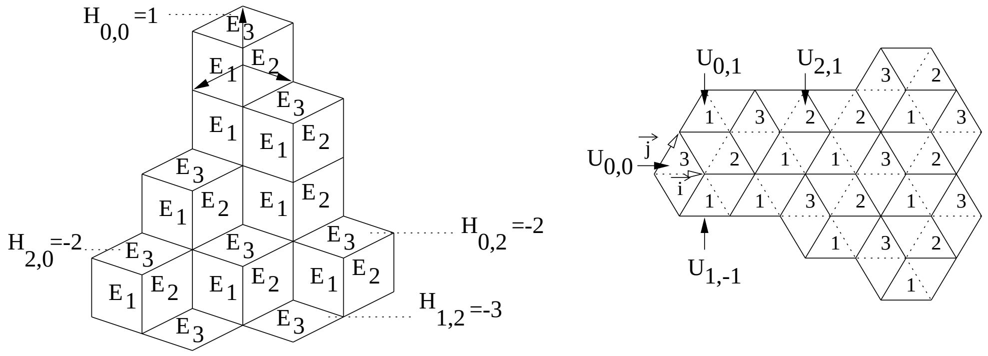
Definition 2 *Let $\overline{\mathcal{S}}$ (respectively $\underline{\mathcal{S}}$) be the set of translates of the fundamental cube with integer vertices that intersect the lower half-space $z < -\alpha x - \beta y + \gamma$ (respectively the upper half-space $z > -\alpha x - \beta y + \gamma$).*

The upper discrete plane $\overline{P}$ *(respectively the* lower discrete plane $\underline{P}$*) is defined as the boundary of $\overline{\mathcal{S}}$ (respectively $\underline{\mathcal{S}}$).*

We will call generically discrete plane *(denoted by P) either $\overline{P}$ or $\underline{P}$, when there is no need to distinguish between them.*

Let $\overline{H}_{p,q} = \lceil -p\alpha - q\beta + \gamma \rceil$, $\underline{H}_{p,q} = \lfloor -p\alpha - q\beta + \gamma \rfloor$, for all $p, q \in \mathbb{Z}$. Note that $\underline{H}_{0,0} = 0$.

Let π be the affine projection on the plane $x + y + z = 0$ according to the direction (1,1,1) and let $\vec{\pi}$ denote the corresponding vectorial projection. Let us endow the plane $x + y + z = 0$ with two bases, according to whether we consider the projections of $\overline{P}$ or $\underline{P}$: consider first for $\overline{P}$ the basis $(O', \vec{i}, \vec{j})$, where $\vec{i} = \vec{\pi}(\vec{e_1} + \vec{e_2})$, $\vec{j} = \vec{\pi}(\vec{e_2})$, and $\vec{OO'} = -\overline{H}_{0,0}\vec{i}$, and second, for $\underline{P}$, the basis $(O, \vec{i}, \vec{j})$.



a) b)

Figure 3: Discrete plane and projection

The projection π has thus the following matrix representation in $(O', \vec{i}, \vec{j})$, $\mathbb{R}^3$ being endowed with its canonical basis $(O, \vec{e_1}, \vec{e_2}, \vec{e_3})$:

$$\pi \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \overline{H}_{0,0} \\ 0 \end{pmatrix},$$

whereas π has the following matrix representation in $(O, \vec{i}, \vec{j})$:

$$\pi \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The discrete plane P is a union of translates of unit square faces. Consider the following square faces

$$E_1 = \{\lambda \vec{e_2} - \mu \vec{e_3} | (\lambda, \mu) \in [0, 1[^2\},$$

$$E_2 = \{\lambda \vec{e_1} - \mu \vec{e_3} | (\lambda, \mu) \in [0, 1[^2\},$$

$$E_3 = \{\lambda \vec{e_1} + \mu \vec{e_2} | (\lambda, \mu) \in [0, 1[^2\}.$$

We call an *upper* (respectively *lower*) *pointed face* of *type* i a set of points

$$\{(p, q, r) + E_i\},$$

(respectively

$$\{(p, q, r) - E_i\}),$$

where $(p, q, r) \in \mathbb{Z}^3$. We say (p, q, r) is the *vertex* of this pointed face. See Figure 4.

Let us prove that the image by the projection π of the discrete plane P generates a **tiling** of the plane by three kinds of diamonds, namely the projections of the faces of type $1, 2$ and 3.

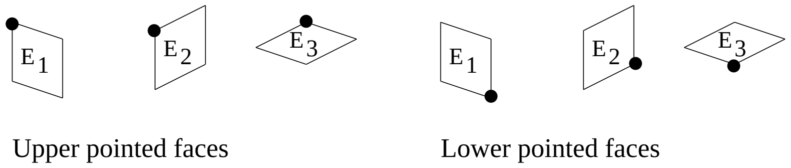


Figure 4: Pointed faces

Lemma 1 *The property stated below applies to both upper and lower faces. The notation $H_{p,q}$ will stand for $\overline{H}_{p,q}$, if one considers upper faces, and for $\underline{H}_{p,q}$, if one considers lower faces.*

The point with coordinates (m, n) is the image by π of the vertex of a face of type k (with $k = 1, 2, 3$) if and only if there exists $(p, q) \in \mathbb{Z}^2$ such that

$$m = p + H_{0,0} - H_{p,q},\ \ n = -p + q,\ \ \textit{if } k = 3,$$

$$p + 1 + H_{0,0} - H_{p,q} \leq m \leq p + H_{0,0} - H_{p,q+1}, n = -p + q,\ \ \textit{if } k = 2,$$

$$p + 1 + H_{0,0} - H_{p,q+1} \leq m \leq p + H_{0,0} - H_{p+1,q+1}, n = -p + q,\ \ \textit{if } k = 1.$$

Furthermore, given any $(m, n) \in \mathbb{Z}^2$, there exists a unique $(p, q) \in \mathbb{Z}^2$ such that (m, n) satisfies one and only one of the above three conditions.

The following corollary is a straightforward consequence of Lemma 1.

Corollary 1 *The projections of the square faces of P tile the plane by three kinds of diamonds being the projection of a face of type E_k, where $k = 1, 2$ or 3. Furthermore, each point of $\mathbb{Z}^2$ is the image of exactly one vertex of one pointed face.*

Proof of Lemma 1 We consider implicitly in the proof the case of the upper discrete plane. The proof works in exactly the same way for the lower discrete plane (the vectorial cubes that we consider below should be replaced by their opposites).

- Consider the faces of type 3. By definition, the cube

  $$\{(p, q, r) + \lambda\vec{e_1} + \mu\vec{e_2} - \nu\vec{e_3},\ 0 \leq \lambda, \mu, \nu \leq 1\}$$

  belongs to $\mathcal{S}$ if and only if $r \leq H_{p,q}$. Hence a face of type E_3 of vertex (p, q, r) belongs to the boundary if and only if $r = H_{p,q}$. The image of $(p, q, H_{p,q})$ by π has coordinates $(p - H_{p,q} + H_{0,0}, -p + q)$.

- Consider the faces of type 2. Let us prove that a point $(p+1, q+1, r)$ is a vertex of a face of type 2 if and only if $H_{p,q+1} + 1 \leq r \leq H_{p,q}$.

  The points with coordinates $(p+1, q+1, H_{p,q} - k)$, with $0 \leq k \leq H_{p,q} - H_{p,q+1} - 1$, are vertices of faces of type E_2. Otherwise, suppose that a face of type 2 with a vertex satisfying $(p+1, q+1, r)$, $H_{p,q+1}+1 \leq r \leq H_{p,q}$, does not belong to the boundary of $\mathcal{S}$. Then the cube

$$\{(p, q+1, r) + \lambda\vec{e_1} + \mu\vec{e_2} - \nu\vec{e_3},\ 0 \leq \lambda, \mu, \nu \leq 1\}$$

would belong to $\mathcal{S}$, and then $r \leq H_{p,q+1}$, which would contradict the assumption on r. In other words, the face of type 3 and vertex $(p, q, H_{p,q})$ is connected to the face of same type and vertex $(p, q+1, H_{p,q+1})$ by $|H_{p,q+1} - H_{p,q}|$ faces of type E_2 (see Figure 5.a).

Conversely, let $(p+1, q+1, r)$ be the vertex of a face of type 2. Suppose that the cube located at the right of this face, i.e.,

$$\{(p, q+1, r) + \lambda\vec{e_1} + \mu\vec{e_2} - \nu\vec{e_3},\ 0 \leq \lambda, \mu, \nu \leq 1\}$$

belongs to $\mathcal{S}$. We thus have $r \leq H_{p,q+1}$. But $H_{p,q} < r$, otherwise the face of type 2 that we consider would not belong to the boundary of $\mathcal{S}$. As $\beta > 1$, $H_{p,q+1} < H_{p,q}$. This implies that this situation cannot occur. Hence $r \geq H_{p,q+1} + 1$ and the cube located at the left of the face we consider, i.e.,

$$\{(p, q, r) + \lambda\vec{e_1} + \mu\vec{e_2} - \nu\vec{e_3},\ 0 \leq \lambda, \mu, \nu \leq 1\}$$

belongs to $\mathcal{S}$, which implies $r \leq H_{p,q}$.

We thus have proved that a point $(p+1, q+1, r)$ is a vertex of a face of type 2 if and only if $H_{p,q+1} + 1 \leq r \leq H_{p,q}$. The projections of these points have coordinates (m, n), with

$$p + 1 + H_{0,0} - H_{p,q} \leq m \leq p + H_{0,0} - H_{p,q+1},\ n = -p + q.$$

- Consider the faces of type 1. We similarly prove that a point with coordinates $(p+1, q+1, r)$ is a vertex of a face of type 1 if and only if $H_{p+1,q+1} + 1 \leq r \leq H_{p,q+1}$, i.e., the face $H_{p,q+1}$ is connected to the face $H_{p+1,q+1}$ by $|H_{p,q+1} - H_{p+1,q+1}|$ faces of type E_1. The projections of these points have coordinates (m, n), with

$$p + 1 + H_{0,0} - H_{p,q+1} \leq m \leq p + H_{0,0} - H_{p+1,q+1}, n = -p + q.$$

As $\alpha > 1$ and $\beta > 1$, we thus have

$$p - H_{p,n+p} < p + 1 - H_{p,n+p} < p - H_{p,n+p+1} < \cdots$$

$$< p + 1 - H_{p,n+p+1} < p - H_{p+1,n+p+1} < p + 1 - H_{p+1,n+p+1}.$$

Hence given any $(m, n) \in \mathbb{Z}^2$, there exists a unique $(p, q) \in \mathbb{Z}^2$ such that (m, n) satisfies one and only one of the three conditions of Lemma 1. ■

Corollary 1 implies that we can **code** the tiling of the plane that we obtain by a two dimensional sequence defined over $\mathbb{Z}^2$ (see Figures 3 and 5). Actually, the projections of the square faces tile the plane by three kinds of diamonds

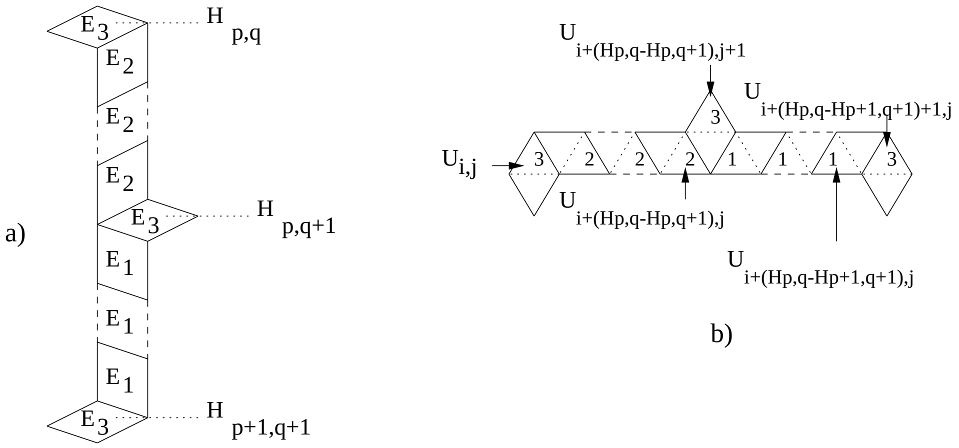


Figure 5: Stairs and two dimensional sequence.

being the projection of a face of type E_k (where $k = 1, 2$ or 3). Each diamond is the juxtaposition of two equilateral triangles: one is called the up triangle (its coordinates in the basis $(O', \vec{i}, \vec{j})$ are (l, m), $(l, m+1)$, $(l+1, m)$), the coordinates of the vertices of the other triangle, which we call the *down triangle*, will depend on the type of the associated projected face. Corollary 1 implies that the centers of the up triangles form a $\mathbb{Z}^2$-lattice (see Figure 3.b). We can thus code this tiling over $\mathbb{Z}^2$ as follows.

Definition 3 *Let $\overline{U} = (U_{m,n})_{(m,n)\in\mathbb{Z}^2}$ (respectively $\underline{U} = (U_{m,n})_{(m,n)\in\mathbb{Z}^2}$) be the sequence that associates with each point with coordinates (m, n) the type of the upper (respectively lower) face whose vertex projects on (m, n), or equivalently which codes each up triangle with coordinates say (m, n), $(m, n+1)$ and $(m+1, n)$ by the index k of the corresponding diamond $\pi(E_k)$.*

The sequence $\overline{U}$ (respectively $\underline{U}$) is called the upper *(respectively the* lower*)* coding *of the plane $\mathcal{P}$.*

We shall use the usual representation for two dimensional sequences: the first index indicates the column number from left to right, whereas the second index n denotes the row number, from bottom to top.

We deduce from Lemma 1 that both two dimensional sequences $\overline{U}$ and $\underline{U}$ satisfy

$$\begin{cases} U_{m,n} = 3 & \Leftrightarrow \exists\, p,\ m = H_{0,0} - H_{p,p+n} + p, \\ U_{m,n} = 2 & \Leftrightarrow \exists\, p,\ H_{0,0} - H_{p,p+n} + p + 1 \leq m \leq H_{0,0} - H_{p,p+n+1} + p, \\ U_{m,n} = 1 & \Leftrightarrow \exists\, p,\ H_{0,0} - H_{p,p+n+1} + p + 1 \leq m \leq H_{0,0} - H_{p+1,p+n+1} + p. \end{cases}$$

This property again holds for both sequences $\overline{U}$ and $\underline{U}$, the notation $H_{p,q}$ standing for $\overline{H}_{p,q}$ or $\underline{H}_{p,q}$, accordingly to the sequence we consider.

We can prove now that the sequences $\overline{U}$ and $\underline{U}$ code a $\mathbb{Z}^2$-action defined by two rotations on the unit circle.

Theorem 1 *Let $\overline{U}$ and $\underline{U}$ be respectively the upper and the lower coding of the plane $\mathcal{P} : z = -\alpha x - \beta y + \gamma$, where $\alpha > 1$, $\beta > 1$, $0 \leq \gamma < 1$. Let $\overline{\gamma} = \gamma$, if $\gamma \neq 0$ and $\overline{\gamma} = 1$, otherwise. Let*

$$\alpha' = \frac{1}{1+\alpha+\beta},\ \beta' = \frac{1+\alpha}{1+\alpha+\beta} = \frac{-\beta}{1+\alpha+\beta},$$

$$\gamma' = \frac{\gamma}{1+\alpha+\beta} \quad \textit{and} \quad \overline{\gamma'} = \frac{\overline{\gamma}}{1+\alpha+\beta}.$$

Both sequences $\overline{U}$ and $\underline{U}$ code the $\mathbb{Z}^2$-action generated by the two rotations of the unit circle $R_{\alpha'}$ and $R_{\beta'}$, of angle respectively α' and β'. More precisely we have

$$\overline{U}_{m,n} = i \Longleftrightarrow R^m_{\alpha'} R^n_{\beta'}(\overline{\gamma'}) \in \overline{\mathcal{I}}_i,$$

with $\overline{\mathcal{I}}_3 =]0, \alpha']$, $\overline{\mathcal{I}}_2 =]\alpha', \alpha' + 1 - \beta']$ and $\overline{\mathcal{I}}_1 =]\alpha' + 1 - \beta', 1]$,

$$\underline{U}_{m,n} = i \Longleftrightarrow R^m_{\alpha'} R^n_{\beta'}(\gamma') \in \underline{\mathcal{I}}_i,$$

with $\underline{\mathcal{I}}_3 = [0, \alpha'[$, $\underline{\mathcal{I}}_2 = [\alpha', \alpha' + 1 - \beta'[$ and $\underline{\mathcal{I}}_1 = [\alpha' + 1 - \beta', 1[$.

Proof Consider the sequence $\overline{U}$.

- Suppose $\overline{U}_{m,n} = 3$. There exists $p \in \mathbb{Z}$ such that

$$m = \overline{H}_{0,0} - \overline{H}_{p,p+n} + p,$$

i.e.,

$$m - \lceil \gamma \rceil = p - \lceil -(\alpha+\beta)p - n\beta + \gamma \rceil.$$

We have

$$(1+\alpha+\beta)p + n\beta < m + \overline{\gamma} \leq (1+\alpha+\beta)p + n\beta + 1,$$

i.e.,

$$p < \frac{m - n\beta + \overline{\gamma}}{1+\alpha+\beta} \leq p + \frac{1}{1+\alpha+\beta},$$

and

$$\frac{m - n\beta + \overline{\gamma}}{1+\alpha+\beta} \in \left]0, \frac{1}{1+\alpha+\beta}\right] \quad \text{modulo } 1.$$

- Suppose $\overline{U}_{m,n} = 2$. There exists $p \in \mathbb{Z}$ such that

$$\overline{H}_{0,0} - \overline{H}_{p,p+n} + p + 1 \leq m \leq \overline{H}_{0,0} - \overline{H}_{p,p+n+1} + p.$$

This is equivalent to

$$\frac{m - n\beta + \overline{\gamma}}{1+\alpha+\beta} \in \left]\frac{1}{\alpha+\beta+1}, \frac{1+\beta}{\alpha+\beta+1}\right] \quad \text{modulo } 1.$$

- Suppose $\overline{U}_{m,n} = 1$. There exists $p \in \mathbb{Z}$ such that

$$p+1-\lceil -(\alpha+\beta)p-(n+1)\beta+\gamma \rceil \leq m - \lceil \gamma \rceil \leq p - \lceil -(\alpha+\beta)p - \alpha - (n+1)\beta + \gamma \rceil,$$

which is equivalent to

$$\frac{m - n\beta + \overline{\gamma}}{1+\alpha+\beta} \in \left]\frac{1+\beta}{\alpha+\beta+1}, 1\right] \quad \text{modulo } 1.$$

The same reasoning holds for $\underline{U}$. ∎

Remarks and notations

- Given any $(m,n) \in \mathbb{Z}^2$, we can give explicitly the coordinates (p,q,r) of the vertex of the face whose projection is coded by $\overline{U}_{m,n}$, say. Let

$$\overline{p} = \left\lceil \frac{m - n\beta + \overline{\gamma}}{1+\alpha+\beta} \right\rceil - 1.$$

If $\overline{U}_{m,n} = 3$, then $p = \overline{p}$, $q = n + p$, $r = \overline{H}_{p,q} = -m + \overline{H}_{0,0} + p$. If $\overline{U}_{m,n} = 2$ or 1, then $p = \overline{p} + 1, q = n + p, r = -m + \overline{H}_{0,0} + p$. A similar result holds for the sequence $\underline{U}$, with $\underline{p} = \left\lfloor \frac{m-n\beta+\gamma}{1+\alpha+\beta} \right\rfloor$.

- This work can be generalized in a higher-dimensional space, either by introducing the notion of a discrete hyperplane (we thus obtain tilings of $\mathbb{R}^{d-1}$), or by considering discrete planes in dimension greater than three (we get tilings of the plane). We will discuss this second point of view in the last section of this paper. For what concerns the first point of view, i.e., codings of discrete hyperplanes, the whole construction and all the results of this section can be extended in a natural way.

- In the sequel we shall work with the sequence $\underline{U}$, that we will denote for simplicity by U. We shall similarly use the notation I_1, I_2, I_3:

$$I_3 = [0, \alpha'[,\ I_2 = [\alpha', \alpha' + 1 - \beta'[,\ I_1 = [\alpha' + 1 - \beta', 1[.$$

Hence, $|I_3| = \frac{1}{1+\alpha+\beta}$, $|I_2| = \frac{\beta}{1+\alpha+\beta}$, $|I_1| = \frac{\alpha}{1+\alpha+\beta}$. We thus have

$$U_{m,n} = i \iff m\alpha' + n\beta' + \gamma' \in I_i.$$

Note that

$$0 < \alpha' < 1/3,\ \alpha' < \beta' < 1,\ \alpha' < 1 - \beta'.$$

- As in the one-dimensional case, one can characterize the case where the sequence U is periodic.

Proposition 2 *The sequence U has a non-zero periodic vector, i.e.,*

$$\exists (a,b) \in \mathbb{Z}^2 - \{(0,0)\},\ \forall (m,n)\ \ U_{m,n} = U_{m+a,n+b},$$

if and only if $1, \alpha', \beta'$ are rationally dependent.

Proof Actually, suppose that $1, \alpha', \beta'$ are rationally independent and that there exists (a,b) a non-zero periodic vector for U. Hence $U_{ka,kb} = U_{0,0}$, for every k, which implies that: $\forall k,\ k(a\alpha' + b\beta') \in [0, \alpha'[$. This is impossible since the sequence $(k(a\alpha' + b\beta'))_{k\in\mathbb{Z}}$ is dense on the unit circle. Conversely, suppose that $1, \alpha', \beta'$ are rationally dependent. There exists $(a,b) \in \mathbb{Z}^2 - \{(0,0)\}$, such that $a\alpha' + b\beta' \in \mathbb{Z}$, and thus, (a,b) is a periodic vector. ∎

3.2 Balance

Let us first deduce the following property of balance from Theorem 1. A (unidimensional) sequence u is called *balanced on the letter i* if for any two factors w and w' of the sequence u of same length, then

$$||w|_i - |w'|_i| \leq 1,$$

where $|w|_i$ denotes the number of occurrences of the letter i in the word w.

Corollary 2 *For every integer n, the sequence $U_n = (U_{m,n})_{m\in\mathbb{Z}}$ is balanced on the letter 3.*

For every integer m, the sequence $U_m = (U_{m,n})_{n\in\mathbb{Z}}$ is balanced on the letter 2.

For every integer m, the sequence $U'_n = (U_{m,n-m})_{m\in\mathbb{Z}}$ is balanced on the letter 1.

Proof Consider, for a fixed n, the sequence $U_n = (U_{m,n})_{m\in\mathbb{Z}}$. Let $pr : \{1,2,3\} \to \{1,3\}$ be the projection defined by $pr(3) = 3$ and $pr(1) = pr(2) = 1$. The sequence $pr(U_n)$ is thus the coding of the point of the unit circle $\gamma' + n\beta'$ under the rotation of angle α' with respect to the partition $\{[0,\alpha'[,[\alpha',1[\}$. Thus the sequence $pr(U_n)$ is balanced on the letter 3 and hence the sequence U_n has the same property. A similar reasoning holds for the other letters. ■

Remark This notion of balance is not completely satisfactory: it might be interesting to get balance properties for rectangular factors and not only for one-dimensional factors. Furthermore, whether one can characterize two dimensional Sturmian sequences via some balance property is an open question.

4 Factors and frequencies

We will assume from now on that $1, \alpha, \beta$ are rationally independent, which implies in particular that α' and β' are irrational numbers.

The aim of this section is to provide a characterization of rectangular factors in terms of intervals of the unit circle. This characterization is classical in the one-dimensional case (see for instance [1, 2]). In Section 4.1 we associate to each rectangular factor a set of points of the unit circle. We prove this set to be connected in Section 4.2. We thus obtain in Section 4.3 a description of the set of rectangular factors of given size in terms of a finite partition of the unit circle.

4.1 Definitions and notation

We call *rectangular factor of the infinite sequence U* a finite array W of consecutive letters of U, say

$$\begin{matrix} w_{1,n} & \cdots & w_{m,n} \\ \vdots & & \vdots \\ w_{1,1} & \cdots & w_{m,1}, \end{matrix}$$

such that there exist k, l satisfying $w_{i,j} = U_{k+i-1,l+j-1}$, with $1 \leq i \leq m$, $1 \leq j \leq n$. We thus say that the factor W has *size* (m, n). Note that here we do not use the usual matrix indexing, but we keep the indexing we previously chose for the representation of two dimensional sequences. Let $L(U, m, n)$ denote the language of rectangular factors of size (m, n) of the sequence U. Let us prove the following.

Lemma 2 *The block $W = [w_{i,j}]$, defined on $\{1, 2, 3\}$ and of size (m, n), is a factor of U if and only if*

$$I(W) := \bigcap_{1\leq i\leq m, 1\leq j\leq n} R_{\alpha'}^{-i+1} R_{\beta'}^{-j+1} I_{w_{i,j}} \neq \emptyset.$$

Moreover, if W is a factor, then given any fixed integer l, there exists an occurrence of W with column index l. The same result holds for the indices of rows.

Proof From Theorem 1, a block $W = [w_{i,j}]$, defined on $\{1, 2, 3\}$ and of size (m, n), is a factor of the sequence U if and only if there exist two integers k, l such that

$$\gamma' + k\alpha' + l\beta' \in I(W) = \bigcap_{1\leq i\leq m, 1\leq j\leq n} R_{\alpha'}^{-i+1} R_{\beta'}^{-j+1} I_{w_{i,j}} \quad \text{modulo } 1.$$

This implies that, if W is a factor, then $I(W) \neq \emptyset$.

Reciprocally, suppose that $I(W) \neq \emptyset$. Then the interior of $I(W)$ is not empty, for $I(W)$ is defined as the intersection of left-closed right-open intervals. Hence for any fixed integer l, there exists k such that $\gamma' + k\alpha' + l\beta' \in I(W)$, since the sequence $(k\alpha')_{k\in\mathbb{Z}}$ is dense on the unit circle. ∎

Example Consider the factors of size $(2, 1)$. As $\alpha' < 1/3$, then $[0, \alpha'[\cap[1 - \alpha', 1[= \emptyset$, i.e., $I_3 \cap R_{\alpha'}^{-1} I_3 = \emptyset$. Hence the word 33 can never be a factor of U.

As $\alpha' < 1 - \beta'$, then

$$I_{(31)} = [0, \alpha'[\cap[1 - \beta', 1 - \alpha'[= \emptyset,$$

$$I_{(32)} = [0, \alpha'[\cap[0, 1 - \beta'[= [0, \alpha'[,$$

$$I_{(22)} = [\alpha', \alpha' + 1 - \beta'[\cap[0, 1 - \beta'[= [\alpha', 1 - \beta'[.$$

As $1 - \alpha' > \alpha' + 1 - \beta'$, then

$$I_{(21)} = [1 - \beta', \alpha' + 1 - \beta'[,$$

$$I_{(11)} = [\alpha' + 1 - \beta', 1 - \alpha'[,$$

$$I_{(13)} = [1 - \alpha', 1[.$$

Furthermore as $\alpha' < 1 - \beta'$, then

$$I_{(12)} = [\alpha' + 1 - \beta'[\cap[0, 1 - \beta'[= \emptyset.$$

Hence if one considers any fixed line, the letter 3 appears as an isolated letter and two successive occurrences of the letter 3 are separated by a range of 2's and then a range of 1's.

4.2 Connectedness

Let us prove now that the sets $I(W)$ are connected. This proof is based on the ideas of [1]. We will use the following remark: if I and J are two left-closed and right-open intervals of the unit circle whose intersection is not connected, then the sum of their lengths is strictly larger than 1 and the ends of I (respectively J) belong to the interior of J (respectively I).

Lemma 3 *Given any rectangular factor $W = [w_{i,j}]$, defined on $\{1,2,3\}$ and of size (m,n), the set*

$$I(W) := \bigcap_{1\leq i\leq m,\ 1\leq j\leq n} R_{\alpha'}^{-i+1} R_{\beta'}^{-j+1} I_{w_{i,j}}$$

defined in Lemma 2 is connected.

Proof Let us prove by induction on $\sup(m,n)$ that given any rectangular factor $W = [w_{i,j}]$ of size (m,n), the set $I(W)$ is connected. The property is obviously true for $m = n = 1$. Suppose that the property holds for a positive integer N such that $\sup(m,n) = N$. Let W be a factor of size $(N+1, n)$, with $n \leq N$. Let

$$I = \bigcap_{1\leq i\leq N,\ 1\leq j\leq n} R_{\alpha'}^{-i+1} R_{\beta'}^{-j+1} I_{w_{i,j}}.$$

The set I is connected by assumption. Fix x in $\{1,2,3\}$. Suppose that $I \cap R_{\alpha'}^{-N} I_x$ is not connected and non-empty. We have $|I| + |I_x| > 1$. This implies that

$$\forall (i,j),\ \text{with}\ 1 \leq i \leq N,\ 1 \leq j \leq n,\ w_{i,j} = x. \tag{1}$$

Otherwise, there would exist (i,j) such that $w_{i,j} = y \neq x$, and then

$$|I| + |I_x| \leq |I_y| + |I_x| < 1.$$

We thus deduce from $2|I_x| \geq |I| + |I_x| > 1$, that $|I_x| > 1/2$, $x \neq 3$ (since $|I_3| = \alpha' < 1/2$) and hence that $|I_x| \leq 1 - \alpha'$.

Let us note $[a,b[= R_{\alpha'}^{-N} I_x$. We have $d(a,b) = |I_x| \leq 1 - \alpha'$, and thus $d(b,a) \geq \alpha'$, where the notation $d(a,b)$ stands for the distance between a and b on the oriented unit circle. The intersection $I \cap [a,b[$ is not connected by assumption. Hence $[b,a]$ is included in the interval I. As $d(b,a) \geq \alpha'$, then $b + \alpha' \in [b,a]$ and hence $b + \alpha'$ belongs to I. Let y be the letter such that the interval I_y follows I_x on the oriented unit circle. Recall that $[a,b[= R_{\alpha'}^{-N} I_x$. Hence $b + \alpha' \in R_{\alpha'}^{-N+1} I_y$. We thus have $R_{\alpha'}^{-N+1} I_y \cap I \neq \emptyset$. But $I \subset R_{\alpha'}^{-N+1} I_{w_{N,1}}$. Hence $w_{N,1} = y \neq x$, which yields the desired contradiction with Equation (1). We have thus proved that $I \cap R_{\alpha'}^{-N} I_x$ is connected.

Let $I' = I \cap R_{\alpha'}^{-N} I_x$. We similarly prove that $I' \cap R_{\alpha'}^{-N} R_{\beta'}^{-1} I_x$, and then $I' \cap (\cap_{2\leq j\leq n}\ R_{\alpha'}^{-N} R_{\beta'}^{-j+1} I_x)$ are connected for any x in $\{1,2,3\}$. Hence the induction property holds for $(N+1, n)$, with $n \leq N$.

Suppose now that $n = N+1$ and $m \leq N+1$. Let $W' = [w_{i,j}]_{1\leq i\leq m, 1\leq j\leq N}$. We thus have $I(W')$ connected. We prove similarly that $I(W') \cap R_{\beta'}^{-N} I_x$ is connected, for any $x \in \{1,2,3\}$. Note that β' may be strictly larger than $1/2$: indeed $\beta' > 1/2$ if and only if $1+\alpha > \beta$. But if $I_x > 1/2$, we have $|I_x| \leq \sup(1-\beta', \beta')$: suppose $\beta' < 1/2$, then $x = 2$ and $|I_2| = 1-\beta'$; suppose $\beta' > 1/2$, then $x = 1$ and $|I_1| < \beta'$. In the first case we use exactly the same proof as previously. In the second case, if we note $[a, b[= R_{\beta'}^{-N} I_x$, we have $d(a,b) < \beta'$ and hence $a + \beta' \in [b, a] \subset I$. We conclude in the same way. Hence we similarly prove that $I(W)$ is connected for any factor of size $(m, N+1)$ $(m \leq N)$ and $(N+1, N+1)$.

We thus have proved that the induction property holds for $N+1$, which completes the proof. ■

More generally, we prove similarly the following.

Lemma 4 *Given any (α'', β''), with $(1, \alpha'', \beta'')$ rationally independent and given any finite partition of the unit circle into intervals of length smaller than* $\sup(\alpha'', 1-\alpha'')$ *and* $\sup(\beta'', 1-\beta'')$, *then any intersection of the form*

$$I = \bigcap_{1 \leq i \leq m, 1 \leq j \leq n} R_{\alpha''}^{-i+1} R_{\beta''}^{-j+1} I_{w_{i,j}},$$

is connected (the block $[w_{i,j}]$ takes its values into the set of indices of the partition).

4.3 Factors

We thus deduce from Lemma 2 and 3 that the intervals $I(W)$, associated to the rectangular factors of size (m, n), are in one-to-one correspondence with the intervals

$$R_{\alpha'}^{-i+1} R_{\beta'}^{-j+1} I_l, 1 \leq i \leq m,\ 1 \leq j \leq n,\ l = 1,2,3.$$

Example The intervals corresponding to the factors of size $(2,1)$ are obtained by putting the extremal points of I_3, I_2, I_1, $R_{\alpha'}^{-1}(I_3)$, $R_{\alpha'}^{-1}(I_2)$, $R_{\alpha'}^{-1}(I_1)$, i.e., the points modulo 1

$$0, \alpha', \alpha' - \beta', -\alpha', -\beta'.$$

In the general case, we have the following.

Lemma 5 *The intervals $I(W)$, associated to the rectangular factors of size (m, n), are obtained by putting on the unit circle the points of the partition $\mathcal{Q}_{m,n}$:*

$$-i\alpha' - j\beta',\ \textit{with}\ -1 \leq i \leq m-1\ \textit{and}\ 0 \leq j \leq n-1,$$

$$-n\beta' - i\alpha', \textit{with}\ -1 \leq i \leq m-2.$$

5 Topological and metric properties

We can thus deduce from Lemmas 2 and 5 the number of rectangular factors of given size (i.e., the rectangular complexity function) by counting points of the corresponding partition (Section 5.1). By considering the lengths of the intervals, we can obtain an upper bound for the number of distinct frequencies for rectangular factors of given size (Section 5.2). Finally, we consider properties of uniform recurrence in Section 5.3.

5.1 Complexity function

Let us associate to the two dimensional sequence U a measure of its "complexity" as follows: let $P(m,n)$ denote the number of distinct rectangular factors of size (m,n) of the sequence U, i.e., $P(m,n) = \mathrm{Card} L(U,m,n)$; the function $(m,n) \to P(m,n)$ is called *(rectangular) complexity* of the sequence U. This quantity is connected to the topological entropy.

The complexity function for triangular and rectangular factors for tilings arising from the projection of a discrete plane is computed in [42]. The proof is based on a notion of geometrical balance and on the study of the combinatorics of patterns. Reveillès also computes in [31] the rectangle complexity function in the rational case.

Let us see how to deduce from Lemma 5 the complexity function for rectangular factors.

Theorem 2 *The complexity in rectangles of the sequence U satisfies*

$$\forall (m,n),\ P(m,n) = mn + m + n.$$

Proof From Lemma 5, the number of factors of size (m,n) of the sequence U is equal to the number of intervals on the unit circle of extremal points (modulo 1) $-i\alpha' - j\beta'$, with $-1 \leq i \leq m-1$ and $0 \leq j \leq n-1$, and $-n\beta' - i\alpha'$, with $-1 \leq i \leq m-2$. There are $mn+m+n$ such points and hence $nm+n+m$ such intervals. ∎

5.2 Frequencies

The *frequency* $f(W)$ of a factor W of the sequence U is defined as the limit, if it exists, of the number of occurrences of this block in the "central" square factor

$$\begin{matrix} U_{-n,n} & \ldots & U_{n,n} \\ \vdots & & \vdots \\ U_{-n,-n} & \ldots & U_{n,-n}, \end{matrix}$$

of the sequence divided by $(2n+1)^2$. Given any interval I of the unit circle, the convergence when n tends towards $+\infty$ of

$$\frac{\mathrm{Card}\{-n \leq i \leq n,\ i\alpha' + \rho \in I\}}{2n+1}$$

towards the length of I is uniform in ρ (in other words, an irrational rotation is *uniquely ergodic*). Hence the frequency of every factor W of U exists and is equal to the length of

$$I(W) = \bigcap_{1\le i\le m, 1\le j\le n} R_{\alpha'}^{-i+1} R_{\beta'}^{-j+1} I_{w_{i,j}}.$$

Therefore the number of frequencies of rectangular factors of the sequence U of size (m,n) is equal to the number of lengths of the partition $\mathcal{Q}_{m,n}$ (defined in Lemma 5), which is given by a two dimensional version of the three distance theorem proved by Geelen and Simpson in [21].

Recall the statement of the *three distance theorem* (see for instance [38, 40, 41] and the survey [2]). For a given α in $]0,1[$, let us place the points $0, \alpha, 2\alpha$, $\ldots, n\alpha$ on the unit circle. These points partition the unit circle into $n+1$ intervals having at most three lengths, one being the sum of the other two. Geelen and Simpson have given the following generalization of this result (see [21]).

Theorem 3 [Geelen and Simpson] *Assume we are given three real numbers α, β, ρ and two positive integers m, n. The set of points*

$$\{i\alpha + j\beta + \rho,\ 0\le i\le m-1,\ 0\le j\le n-1\}$$

partitions the unit circle into intervals having at most $\min\{m,n\}+3$ *lengths.*

We thus deduce the following result on frequencies.

Proposition 3 *The frequencies of rectangular factors of size (m,n) of the sequence U take at most* $\min\{m,n\}+5$ *values.*

Proof The set of points

$$\{-i\alpha' - j\beta',\ -1\le i\le m-1,\ 0\le j\le n\}$$

partitions the unit circle into intervals having at most $\min\{m,n\}+4$ lengths. The number of frequencies of rectangular factors of the sequence U of size (m,n) is equal to the number of lengths of the partition $\mathcal{Q}_{m,n}$. The point $-(m-1)\alpha'-n\beta'$ does not belong to $\mathcal{Q}_{m,n}$, hence one more length might appear.∎

Remarks

- Whether this bound is sharp is an open question.
- In the Sturmian case, there are at most three frequencies, when one considers factors of given length (see [6]); this result corresponds to the three distance theorem. More generally, if one codes the orbit of a point of the unit circle under an irrational rotation with respect to a partition into two intervals, then there are at most 5 frequencies for the factors of given length. An expression of these five frequencies is given in [9] using an algorithm of approximation (modulo 1) of β by the points $k\alpha$.

5.3 Uniform recurrence

A sequence is said to be *uniformly recurrent* if for every n, there exists an integer N such that every square factor of size (N, N) contains every square factor of size (n, n).

Proposition 4 *The sequence U is uniformly recurrent.*

Proof The uniform recurrence is a direct consequence of the three gap theorem (see [37]): let δ be an irrational number and I be an interval of the unit circle; the gaps between the successive integers j such that $\{\delta j\} \in I$ take at most three values which only depend on the length of I. Let $l(n)$ denote the smallest of the at most $n+5$ lengths of the intervals corresponding to the square factors of U of size (n, n). Let $g(\alpha', n)$ be the length of the largest gap obtained when the three gap theorem is applied using the irrational number α' and an interval of length $l(n)$. Then every factor of size (n, n) appears in every factor of size $(g(\alpha', n) + n - 1, n)$ (and thus in every square factor) of size $(g(\alpha', n) + n - 1, g(\alpha', n) + n - 1)$. Note that the same reasoning applies when exchanging α' and β'. ■

From an ergodic point of view, this corresponds to the minimality of the $\mathbb{Z}^2$-action.

Actually we have proved a stronger property than uniform recurrence: for every n, there exists an integer N such that every square factor of size (n, n) in $L(U, n, n)$ appears in every rectangular factor of size (N, n), which implies the following proposition.

Proposition 5 *For any integer k, the language of rectangular factors of the sequences constructed with k consecutive rows of the form*

$$((U_{n,i}), (U_{n,i+1}), \cdots, (U_{n,i+k-1}))_{n \in \mathbb{Z}}$$

does not depend on the index i.

This property of "strong" uniform recurrence has the following geometric interpretations.

Proposition 6 *Two two dimensional Sturmian sequences coding two discrete planes with the same totally irrational normal have the same language of rectangular factors.*

Proof The quantity $\bigcap_{1 \leq i \leq l, 1 \leq j \leq k} R_{\alpha'}^{-i+1} R_{\beta'}^{-j+1} I_{w_{i,j}}$ is independent of the height γ' of the plane. Since we work with semi-intervals, then the fact that $\bigcap_{1 \leq i \leq l, 1 \leq j \leq k} R_{\alpha'}^{-i+1} R_{\beta'}^{-j+1} I_{w_{i,j}}$ is non-empty does not depend on the sense of the partition. Therefore, the sequences $\overline{U}$ and $\underline{U}$ (that is the lower and the upper codings introduced in Section 3) have also the same set of factors.

Proposition 7 *Two discrete planes with the same totally irrational normal vector have the same finite patterns of faces.*

Proof Let F be a finite pattern of faces (not necessarily connected) appearing in the sequence U for a given plane of equation $z = -\alpha x - \beta y + \gamma$. There exists a rectangular word W, which is a factor of U, and such that F appears in W. Consider a sequence U' given by a discrete plane with same normal vector. By the preceding proposition, W is a factor of U'. Then F appears in U'. ∎

6 Minimal complexity

In the one-dimensional case, a sequence satisfying $p(n) \leq n$ for some n, is ultimately periodic (see [29], and also [16]). Hence Sturmian sequences are the sequences of lowest complexity among the non-ultimately periodic sequences. In the higher-dimensional case, no such results have been obtained: which complexity functions are realisable, and whether or not there is a notion of minimal complexity for two dimensional sequences, are both open questions. The aim of this section is to provide an example of sequence of complexity $mn + n$, obtained by a projection of the sequence U.

Definition 4 *Consider the letter-to-letter projection $pr : \{1,2,3\} \to \{1,3\}$ defined by $pr(1) = pr(2) = 1$ and $pr(3) = 3$. Let $V = (V_{m,n})_{(m,n)\in\mathbb{Z}^2}$ be the image by the projection pr of the sequence U. We thus have*

- $V_{m,n} = 3 \iff 0 \leq \{m\alpha' + n\beta' + \gamma'\} < \alpha'$,
- $V_{m,n} = 1 \iff \alpha' \leq \{m\alpha' + n\beta' + \gamma'\} < 1$.

The sequences in rows are Sturmian sequences that have the same language (corresponding to the angle α'), whereas the sequences in columns are binary codings with respect to the partition $\{[0,\alpha'[, [\alpha',1[\}$ of the rotation of angle β'.

Lemma 6 *There exist m_0, n_0 such that the complexity of the sequence V is equal to $mn + n$, whenever $m > m_0$* **or** *$n > n_0$.*

Proof The sets $I(W)$ corresponding (as defined in Section 4.1) to the rectangular factors of size (m,n) are bounded by the points $-i\alpha' - j\beta'$, with $-1 \leq i \leq m-1$, $0 \leq j \leq n-1$. These sets are not connected for the first values of m, n because of the following relationship between α' and β': $\alpha' < 1-\beta'$ and $1-\beta' < 1-\alpha'$. In fact, consider the factor $W = \begin{matrix} 1 \\ 1 \end{matrix}$. We thus have

$$I(W) = [\alpha', 1[\cap[\alpha' + 1 - \beta', 1 - \beta'[= [\alpha' + 1 - \beta', 1[\cup[\alpha, 1 - \beta'[.$$

However, by using the same argument as in Section 4.2, we prove the following: if $I(W)$ is not connected for a given rectangular factor $W = [w_{i,j}]$, then there exists x such that: $\forall (i,j)$, $w_{i,j} = x$. But the set of integers n such that x^n is a factor of any Sturmian sequence of angle α is bounded: such a bound can be explicitly given with respect to α (see [2]). The same result holds by considering the sequences in columns, i.e., the binary codings with respect to the partition $\{[0,\alpha'[, [\alpha',1[\}$ of the rotation of angle β'. Hence if n or m are large enough, then the sets $I(W)$ are connected. Hence there exist positive integers m_0 and n_0 such that the complexity of the sequence V is equal to $mn+n$, whenever $m > m_0$ or $n > n_0$. ∎

Following this idea, let us construct a sequence of complexity satisfying $P(m,n) = mn+n$, **for every** (m,n).

Proposition 8 *Let $\alpha'', \beta'' \in]0,1[$ be such that $(1, \alpha'', \beta'')$ is totally irrational and* $\sup(\alpha'', 1-\alpha'') \leq \sup(\beta'', 1-\beta'')$. *Let $V = (V_{m,n})_{(m,n)\in\mathbb{Z}^2}$ be the sequence defined as follows:*

- $V_{m,n} = 3 \iff m\alpha'' + n\beta'' \in [0, \alpha''[$ *modulo* 1,
- $V_{m,n} = 1 \iff m\alpha'' + n\beta'' \in [\alpha'', 1[$ *modulo* 1.

The complexity of the sequence V satisfies:

$$\forall (m,n),\ P(m,n) = mn+n.$$

Furthermore, the sequence V is uniformly recurrent and has no periodic rational direction: there do not exist $(a,b) \in \mathbb{Z}^2 - \{(0,0)\}$ such that

$$\forall (m,n) \in \mathbb{Z}^2,\ V_{m,n} = V_{m+a,n+b}.$$

Proof Following Section 4.2 (Lemma 4), the sets $I(W)$ are easily seen to be connected under the assumptions of the proposition. This yields the expression of the complexity function.

The uniform recurrence and the non-periodicity of the two dimensional sequence V can be proved in the same way as that of U.

∎

We thus obtain an example of a uniformly recurrent sequence with no periodic rational direction and of low complexity ($\forall (m,n),\ P(m,n) = mn+n$). For more results on these sequences, see [7, 8]. No other example of a uniformly recurrent sequence with no periodic rational direction and of lower complexity seems to be known.

Let us end this paper by recalling the following conjecture: let V be a two dimensional sequence defined on the alphabet $\mathcal{A}$; if there exist two positive integers n_0, m_0 such that

$$P(m_0, n_0) \leq m_0 n_0,$$

then the sequence V has a periodic rational direction. The converse is false: take a binary unidimensional sequence of maximal complexity ($p(n) = 2^n$, for every n). One can construct such a sequence $C = (C_n)_{n\in\mathbb{N}}$ by using the Champernowne construction (see for instance [15]), i.e., by concatenating the base 2 expansions of the absolute value of the integers. Hence all possible combinations of the digits 0 and 1 appear in the sequence C and: $\forall n,\ p(n) = 2^n$. The double-sequence $(C(m,n))_{(m,n)\in\mathbb{N}^2}$ defined by $C(m,n) = C(0, m+n)$ has the periodic direction $(-1,1)$ and its complexity satisfies: $\forall (m,n),\ P(m,n) = 2^{m+n-1}$.

Note that Sander and Tijdeman have proved in [35] the conjecture for factors of size $(2, n)$ or $(n, 2)$, i.e., if there exists n such that $P(2, n) \leq 2n$ or $P(n, 2) \leq 2n$, then the sequence is periodic. They state a more general conjecture in [33, 34, 35], by extending the notion of complexity. Furthermore a counterexample to the corresponding conjecture for k-dimensional sequences with $k \geq 3$ is given in [34]. Epifanio, Koskas and Mignosi prove in [19] a weakened version of the conjecture: if there exist (m_0, n_0) such that $P(m_0, n_0) \leq \alpha m_0 n_0$, with $\alpha = 1/100$, then the sequence is periodic. The limiting case $mn + 1$ has been exhaustively described by Cassaigne in [14], see also [7] for a study of two dimensional sequences of which language of factors in line is the language of a recurrent Sturmian sequence.

7 Construction of double sequences associated with tilings in higher dimensions

Consider now the extension of these results to discrete planes (not hyperplanes) in dimension greater than three. Indeed, the “cut and project” method (see for instance [36]) we have considered here provides a natural way first, to construct tilings of the plane with diamonds, and second, to code them by two dimensional sequences over a finite alphabet.

Consider a plane $\mathcal{P}$ (i.e., an affine space of dimension 2) in $\mathbb{R}^d$ (with $d \geq 4$). As previously, the discrete plane P associated to $\mathcal{P}$ is defined as the (upper or lower) surface of the union of all unit hyper-cubes, with vertices at integer lattice points, which intersect the plane $\mathcal{P}$. Consider a projection π onto an affine plane, for instance, the orthogonal projection on $\mathcal{P}$. The projection of the points with integer coordinates of the discrete plane P determines the vertices of the tilings. For example, we show in Figure 6 [1] a part of a tiling arising from a discrete plane in $\mathbb{R}^4$ and in Figure 7, a tiling arising from a discrete plane in $\mathbb{R}^5$: this tiling is a Penrose Tiling.

The next step is to construct an explicit two dimensional sequence associated to such a tiling. The idea is to parametrize the discrete plane by a lattice. We have defined in $\mathbb{R}^3$ pointed faces in such a way that the discrete plane is partitioned into these pointed faces. The projection π maps the vertices of the discrete plane (i.e., the integral points) in a one-to-one way onto a $\mathbb{Z}^2$-lattice Γ (see Corollary 1): each point of Γ is the projection of a distinguished vertex of a face of determined type. In this sense, one can say that the two dimensional sequences associated to these tilings are quite natural.

In the higher-dimensional case, we can also map the tiling onto a $\mathbb{Z}^2$-lattice. Consider for instance the construction for $\mathbb{R}^4$ and $\mathbb{R}^5$. First, note that the projections of the faces in a discrete plane in dimension $\mathbb{R}^d$ tile a $2d-$gone, as the projection of a unit hyper-cube in $\mathbb{R}^d$. In consequence, it is sufficient [18] to map the vertices of the $2d-$gone onto integer points of $\mathbb{Z}^2$, in order to map the tiling onto a $\mathbb{Z}^2$-lattice (see Figure 8).

[1] These figures have been realized thanks to QuasiTiler (http://www.geom.umn.edu/apps/quasitiler/about.html).

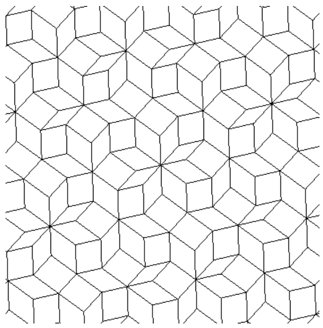


Figure 6: Projection of a discrete plane in $\mathbb{R}^4$

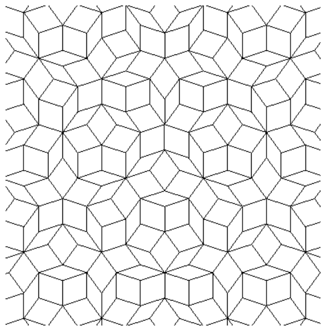


Figure 7: Projection of a discrete plane in $\mathbb{R}^5$

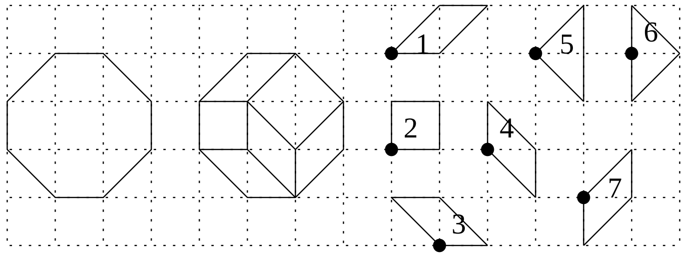


Figure 8: An octogone mapped onto a $\mathbb{Z}^2$ lattice

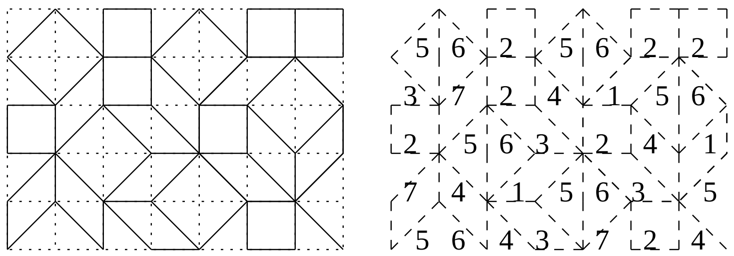


Figure 9: Seven pointed faces

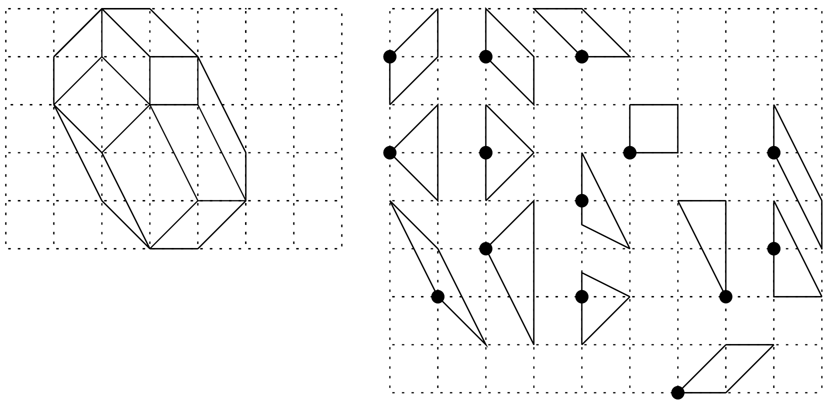


Figure 10: Fourteen pointed faces

For $d = 4$, all the vertices are mapped onto integer points. The only trouble is that there exists an integer point in the interior of one of the faces. To solve it, we simply cut this face into two parts (pointed faces 5 and 6 on Figure 8). We thus find a two dimensional sequence defined on a seven-letter alphabet associated to the discrete plane in $\mathbb{R}^4$ (see Figure 9).

For $d = 5$, (see Figure 10) all the vertices are mapped onto integer points. There are two integer points which belong respectively to the interior of two faces and two integer points which belong to the interior of the same face. We simply cut the first two faces into two parts and the third one into three parts. We thus find, for a discrete plane in $\mathbb{R}^5$ (i.e., a Penrose tiling), a two dimensional sequence defined on a fourteen-letter alphabet.

The notion of dynamical system associated with Penrose tilings has been introduced by Robinson in [32]. See also [23] for the notion of Sturmian dynamical systems. We guess that it is possible to associate as previously to the tilings defined here a rotation in a torus of appropriate dimension.

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