Local configurations in a discrete plane.

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Abstract : *We study the number of local configurations in a discrete plane. We convert this problem into a computation of a double sequence complexity. We compute the number $C(n, m)$ of distinct $n \times m$ patterns appearing in a discrete plane. We show that $C(n, m) = nm$ for all n and m positive integers. The coding of this sequence by a $\mathbb{Z}^2$-action on the unidimensional torus gives information about the structure of a discrete plane. Furthermore, this sequence is a generalized Rote sequence with complexity $P(n, m) = 2nm$ for all n and m positive integers and with a symmetric complementary language for rectangular words.*

Key-words: Discrete planes, double sequence complexity, plane partitions, symmetric complementary language, generalization of Rote and Sturmian sequences.

1 Introduction

In this article, we use the notion of complexity for a double sequence to study local configurations in a discrete plane. The complexity function $p(n)$ for a sequence in a finite alphabet is defined from $\mathbb{N}$ to $\mathbb{N}$ and gives the number of distinct words of length n appearing in the sequence. The survey [1] for sequences with one index and values in a finite alphabet is a good reference with an extensive bibliography. We extend the notion of complexity to double sequences in a finite alphabet. The complexity function $P(n, m)$ is defined from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. Consider a sequence $U = (U_{n,m})_{(n,m)\in\mathbb{Z}^2}$ and the language of blocks $L(U,n,m)$ which is the set of all the blocks $n \times m$ appearing in the sequence. We define the complexity function as $P(n,m) =$ Card $L(U,n,m)$. This complexity measures how "complicated the sequence is" and it is related to the topological entropy.

The study of double sequences is recent. Salon studies double sequences but in terms of substitutions in two dimensions and formal series with two indices ([12]). In [4], Berthé and Vuillon give the complexity of a tiling with lozenges arising from a discrete plane (i.e. the discretization of a plane in $\mathbb{R}^3$: the upper profile of the set of all unit cube with vertices in integer coordinate having a non-empty intersection with the plane). The method uses a double sequence in a three-letters alphabet and a coding in the one-dimensional torus. They approach the problem of minimal complexity for double sequences. In [2], Allouche and Berthé give the complexity of the Pascal triangle modulo 2:

$$P(n,m) = n^2 + m^2 + 2nm - 3n - 3m + 4, \ \forall (n,m) \in \mathbb{N}^2.$$

Furthermore, they find an explicit formula for the complexity of the Pascal triangle modulo d when d is a prime number, and bounds in the other cases.

The structure of discrete planes is intimately related to multi-dimensional continued fractions. In particular, Ito showed that the discrete plane is generated by substitutions on faces using the Jacobi-Perron algorithm (see [9]). The discrete plane can also be seen by a natural generalization of the Sturmian sequences (see [4],[5],[6] and [14]).

In this article, we study discrete planes by counting the number of local configurations. We convert this problem into the computation of double sequence complexity. This approach may have applications in computer science in the field of discrete plane recognition (see [8] and [10]). In particular our coding by two rotations in the one-dimensional torus could be a good tool for computer scientists.

Let a plane in $\mathbb{R}^3$ be defined by $z = -\alpha x - \beta y + \gamma$. We construct a discrete plane by a discretization of the plane. The discretization is given by three square faces oriented according to the three coordinate planes. Let $(H_{i,j})_{(i,j)\in\mathbb{Z}^2}$ be the infinite array giving the height of horizontal faces. We consider the finite pattern $(n \times m)$ which is called plane partition by combinatorialists (see [13]):

$$PP(n,m;i,j) = \begin{array}{llll} H_{i-n+1,j-m+1} - H_{i,j} & H_{i-n+1,j-m+2} - H_{i,j} & \cdots & H_{i-n+1,j} - H_{i,j} \\ H_{i-n+2,j-m+1} - H_{i,j} & H_{i-n+2,j-m+2} - H_{i,j} & \cdots & H_{i-n+2,j} - H_{i,j} \\ \vdots & \vdots & & \vdots \\ H_{i,j-m+1} - H_{i,j} & H_{i,j-m+2} - H_{i,j} & \cdots & H_{i,j} - H_{i,j}. \end{array}$$

The goal of this article is to show that the number of elements of the set

$$\{PP(n,m;i,j) | (i,j) \in \mathbb{Z}^2\}$$

(i.e. the number of distinct $n \times m$ patterns appearing in the discrete plane) is

$$C(n,m) = nm, \ \forall (n,m) \in \mathbb{N}^2.$$

The structure of the article is the following: in the second section, we give a brief exposition of the construction of the discrete plane. In the third section, we construct a sequence

$$(U_{i,j} = H_{i,j} \text{ mod } 2)_{(i,j) \in \mathbb{Z}^2}$$

with double indices and values in the alphabet $\{0,1\}$ associated to the discrete plane and a coding by a $\mathbb{Z}^2$-action of the one-dimensional torus. In section four, we compute the complexity of such a sequence (i.e. the number of $n \times m$ rectangular words appearing in the sequence) which is

$$P(n,m) = 2nm, \ \forall (n,m) \in \mathbb{N}^2.$$

In section five, we introduce a method which associates a plane partition to a rectangular word in the alphabet $\{0,1\}$ appearing in the sequence and the converse. We establish the number of distinct $n \times m$ plane partitions appearing in the discrete plane. In section six, we study the language of the double sequence. In the last section, we give the generalization of the results to all dimensions.

2 Construction of the discrete plane.

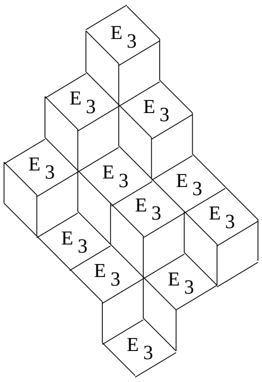
Consider the plane $\mathcal{P}$ in $\mathbb{R}^3 : z = -\alpha x - \beta y + \gamma$ with α and β positive irrationals and with the vector $(1, \alpha, \beta)$ totally irrational (i.e. $l + m\alpha + n\beta = 0$ for $(l,m,n) \in \mathbb{Z}^3 \Rightarrow (l,m,n) = (0,0,0)$). Consider the following square faces:

$$E_1 = \{\lambda \vec{e_2} + \mu \vec{e_3} | (\lambda, \mu) \in [0,1]\};$$

$$E_2 = \{\lambda \vec{e_1} + \mu \vec{e_3} | (\lambda, \mu) \in [0,1]\};$$

$$E_3 = \{\lambda \vec{e_1} + \mu \vec{e_2} | (\lambda, \mu) \in [0,1]\},$$

where $(O, \vec{e_1}, \vec{e_2}, \vec{e_3})$ is a orthonormal basis of $\mathbb{R}^3$.



2	1	0	0
1	0	0	-1
0	-1	-1	-2

Figure 1: Discrete plane and height array.

We associate a discrete plane to the plane $\mathcal{P}$ as follows: let $\mathcal{S}$ be the set of translates of the unit cube interior with integer vertices that $\mathcal{P}$ intersects. The discrete plane is defined as the upper side of the boundary of $\mathcal{S}$. The discretization is given by the following method : we compute for all integers n and m the height $H_{n,m}$ of the face E_3 with coordinates $(n, m, H_{n,m})$ given the position of the face origin (i.e. $\lambda = 0$ and $\mu = 0$ in the definition of E_3). We note E_3 $(n, m, H_{n,m})$ the face E_3 with coordinates $(n, m, H_{n,m})$.

We have for all n and m integers:

$$H_{n,m} = \lceil -n\alpha - m\beta + \gamma \rceil.$$

In the lattice $\mathbb{Z}^3$, we place for all $(n, m) \in \mathbb{Z}^2$, a face of type E_3 in position $(n, m, H_{n,m})$. For fixed n, we connect each face of type E_3 $(n, m, H_{n,m})$ $m \in \mathbb{Z}$ to the face E_3 $(n, m+1, H_{n,m+1})$ by $|H_{n,m+1} - H_{n,m}|$ faces of type E_2. For fixed m, we connect each face of type E_3 $(n, m, H_{n,m})$ $n \in \mathbb{Z}$, to the face E_3 $(n+1, m, H_{n+1,m})$ by $|H_{n+1,m} - H_{n,m}|$ faces of type E_1. This object is the discrete plane associated to the plane $\mathcal{P}$. It has been studied by several authors [4],[9] and [14]. For discussion of recognition of discrete plane, Françon and Reveillès consider finite parts of a discrete plane (see [8] and [10]).

3 Construction of the double sequences.

The coordinates of all horizontal faces are given by the triplet $(i, j, H_{i,j})$ on the lattice $\mathbb{Z}^3$. Let H be the infinite array of integers given by the height array $H = (H_{i,j})_{(i,j)\in\mathbb{Z}^2}$ (see Figure 1). To study the plane partitions appearing in the discrete plane, we look at the sequence $U = (U_{i,j})_{(i,j)\in\mathbb{Z}^2}$ defined by:

$$U_{i,j} = H_{i,j} \bmod 2, \ \forall (i,j) \in \mathbb{Z}^2.$$

Let R_α be the rotation of angle α on $[0,1[$ defined by:

$$R_\alpha(x) = \{x + \alpha\},$$

where $\{\ \}$ denotes the usual fractional part.

Theorem 3.1 *Let $\mathcal{P}$ be a plane $z = -\alpha x - \beta y + \gamma$ with α and β positive reals and $(1, \alpha, \beta)$ totally irrational. Let U be the sequence constructed by $U_{i,j} = H_{i,j}$ mod 2, for all $(i,j) \in \mathbb{Z}^2$. Then the sequence $(U_{i,j})_{(i,j)\in\mathbb{Z}^2}$ is given by the coding of a rotation on the unit circle:*

$$U_{i,j} = \mathcal{I}\left(R^i_{-\frac{\alpha}{2}} R^j_{-\frac{\beta}{2}}\left(\frac{\gamma}{2}\right)\right),$$

where $\mathcal{I}(x) = 1$ if $x \in]0, \frac{1}{2}]$ and $\mathcal{I}(x) = 0$ if $x \in]\frac{1}{2}, 1[\cup \{0\}$.

<u>Proof</u>. We consider the sequence $H_{i,j}$ mod 2. For each $(i,j) \in \mathbb{Z}^2$, we have to study the parity of $H_{i,j}$. Let $(i,j) \in \mathbb{Z}^2$. By definition of the sequence, if there exists an integer n such that $H_{i,j} = 2n+1$ then $U_{i,j} = 1$ otherwise $U_{i,j} = 0$. Furthermore $H_{i,j}$ is the height of a face E_3 and we have $H_{i,j} = \lceil -i\alpha - j\beta + \gamma \rceil$. Thus if there exists an integer n such that $-i\alpha - j\beta + \gamma \in]2n, 2n+1]$ then $U_{i,j} = 1$ else $U_{i,j} = 0$.

If $\{\frac{-i\alpha - j\beta + \gamma}{2}\} \in]0, \frac{1}{2}]$ then $U_{i,j} = 1$ or if $\{\frac{-i\alpha - j\beta + \gamma}{2}\} \in]\frac{1}{2}, 1[\cup \{0\}$ then $U_{i,j} = 0$.

In terms of rotation

$$U_{i,j} = \mathcal{I}\left(R^i_{-\frac{\alpha}{2}} R^j_{-\frac{\beta}{2}}\left(\frac{\gamma}{2}\right)\right),$$

where $\mathcal{I}(x) = 1$ if $x \in]0, \frac{1}{2}]$ and $\mathcal{I}(x) = 0$ if $x \in]\frac{1}{2}, 1[\cup \{0\}$. $\diamondsuit$

This theorem gives informations about the heights of a discrete plane modulo 2: this is a coding of a $\mathbb{Z}^2$ action on the one-dimensional torus with the partition $I_1 =]0, \frac{1}{2}]$ and $I_0 =]\frac{1}{2}, 1]$. It also gives a characterization of the discrete plane.

4 Words, languages and complexity.

For all the following sections, we fix α, β, γ and a sequence U arising from a discrete plane associated to a given plane $\mathcal{P}$ with totally irrational normal. Let $\alpha' = \frac{\alpha}{2}$, $\beta' = \frac{\beta}{2}$ and $\gamma' = \frac{\gamma}{2}$. A word w defined on $\{0,1\}^{n\times m}$,by

$$w = \begin{matrix} w_{n,1} & \dots & w_{n,m} \\ \vdots & & \vdots \\ w_{1,1} & \dots & w_{1,m} \end{matrix}$$

is defined to be a factor of the sequence $U = (U_{i,j})_{(i,j)\in\mathbb{Z}^2}$ if and only if there exists integers k, l such that

$$\{k\alpha' + l\beta' + \gamma'\} \in I(w),$$

where

$$I(w) = \bigcap_{1\leq i\leq n, 1\leq j\leq m} R_{\alpha'}^{i-1} R_{\beta'}^{j-1} I_{w_{i,j}},$$

with $I_1 =]0, \frac{1}{2}]$ and $I_0 =]\frac{1}{2}, 1]$ (see [7]).

Indeed, if w is a factor of the sequence U, we have x and y such that for $1 \leq i \leq n$ and $1 \leq j \leq m$, $w_{i,j} = U_{x+i-1,y+j-1}$. Then $w_{i,j} = \mathcal{I}(R_{-\alpha'}^{i-1+x} R_{-\beta'}^{j-1+y}(\gamma')) = \mathcal{I}(R_{-\alpha'}^{i-1} R_{-\beta'}^{j-1}(\rho))$, where $\rho = R_{-\alpha'}^{x} R_{-\beta'}^{y}(\gamma')$. In other words: for $1 \leq i \leq n$ and $1 \leq j \leq m$, $w_{i,j} = U_{x+i-1,y+j-1}$ if and only if $R_{-\alpha'}^{i-1} R_{-\beta'}^{j-1}(\rho) \in I_{w_{i,j}}$. That is for $1 \leq i \leq n$ and $1 \leq j \leq m$, $w_{i,j} = U_{x+i,y+j}$ if and only if $\rho \in \bigcap_{1\leq i\leq n, 1\leq j\leq m} R_{\alpha'}^{i-1} R_{\beta'}^{j-1} I_{w_{i,j}}$.

$I(w)$ is a segment on the unit circle associated to the word w. As α is irrational, the sequence $(\{\gamma' + k\alpha'\})$ is dense in the unit circle, which implies that w is a factor of U if and only if $I(w) \neq \emptyset$.

Examples: The 1×1 rectangular words 0 and 1 are factors of the double sequence because $I(0) = I_0$ and $I(1) = I_1$ are non empty intervals. Consider the 1×2 rectangular words and let $\beta' < 1/2$ $I(w) = \bigcap_{1\leq j\leq m} R_{\beta'}^{j-1} I_{w_{1,j}}$. $I(11) =]\beta', 1/2]$, $I(00) =]\beta'+1/2, 1]$, $I(01) =]1/2, 1/2+\beta']$ and $I(10) =]0, \beta']$. All these four intervals are non empty then 11, 00, 01 and 10 are factors of the double sequence U. An other way to build these intervals for 1×2 rectangular words is to put extremal points on the unit circle: the points $\{j\beta'\}$ with $0 \leq j \leq 1$ and $\{1/2 + j\beta'\}$ with $0 \leq j \leq 1$. More generally, the intervals associated to $1 \times m$ rectangular words are built by putting the following extremal points on the unit circle: the points $\{j\beta'\}$ with $0 \leq j \leq m-1$ and $\{1/2 + j\beta'\}$ with $0 \leq j \leq m-1$.

Let $L(U, n, m) = \{W \in \{0,1\}^{n \times m} | I(W) \neq \emptyset\}$ be the language of $n \times m$ rectangular words of the sequence U and the language of the sequence U is defined to be the union of the languages of $n \times m$ for all n and m positive integers. We define the complexity function for rectangular words as the number of distinct rectangular words appearing in the tiling. The complexity function of U is the function $P : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ defined by:

$$P(n, m) = \text{ Card } L(U, n, m).$$

Theorem 4.1 *The complexity of the sequence U is $P(n, m) = 2nm$, $\forall (n, m) \in \mathbb{N}^2$*

Proof. Consider a $n \times m$ rectangular word w. Thus we have w is a factor of the sequence if and only if $I(w) \neq \emptyset$. Then the number of distinct $n \times m$ words is equal to the number of intervals on the unit circle

$$\bigcap_{1 \leq i \leq n, 1 \leq j \leq m} R_{\alpha'}^{i-1} R_{\beta'}^{j-1} I_{w_{i,j}}.$$

The extremal points of the intervals are given by the points $\{i\alpha' + j\beta'\}$ with $0 \leq i \leq n-1, 0 \leq j \leq m-1$ and $\{\frac{1}{2} + i\alpha' + j\beta'\}$ with $0 \leq i \leq n-1, 0 \leq j \leq m-1$. There are $2nm$ such points. As the vector $(1, \alpha, \beta)$ is totally irrational, all these points are distinct. Thus there exists $2nm$ distinct intervals. That is, we have $2nm$ intervals corresponding to distinct patterns $n \times m$. Therefore the complexity of the tiling is $P(n, m) = 2nm$ for all $(n, m) \in \mathbb{N}^2$. $\diamondsuit$

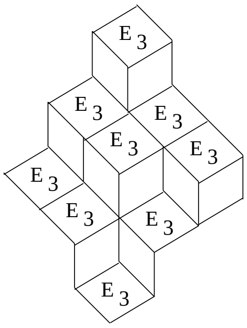
This is a generalization of Rote sequences [11]. Indeed, he studied the sequences in finite alphabet with complexity $p(n) = 2n$ for all n positive integer. As $P(1, m) = 2m$ for all $m \in \mathbb{N}$ and $P(n, 1) = 2n$ for all $n \in \mathbb{N}$ and by the coding of the $\mathbb{Z}^2$-action, we find the Rote sequences for each row and column.

5 Plane partitions and $n \times m$ patterns.

5.1 Plane partitions

A plane partition of an integer m is an array of $l \times k$ integers (l rows and k columns) $\{z_{i,j}\}$ satisfying:

$$0 \leq z_{i,j} \leq m, \ z_{i,j} \leq z_{i+1,j} \text{ and } z_{i,j} \leq z_{i,j+1}.$$



3	2	2
2	2	1
1	1	0

1	0	0
0	0	1
1	1	0

Figure 2: 3×3 pattern, plane partition and binary word.

The number of plane partitions $G_{klm}(1)$ for fixed k, l and m was found by Mac Mahon and is given by

$$G_{klm}(x) = \frac{F_{k+l+m}(x)\ F_k(x)\ F_l(x)\ F_m(x)}{F_{k+l}(x)\ F_{l+m}(x)\ F_{k+m}(x)}$$

where $F_n(x) = (1-x)^{n-1}(1-x^2)^{n-2}\cdots(1-x^{n-1})$.

There exists a three-dimensional representation of a plane partition. We replace all integer $z_{i,j}$ by a tower of cubes of height $z_{i,j}$.

In our case, we would like to count the number of plane partitions arising from a discrete plane:

$$PP(n,m;i,j) = \begin{array}{llll} H_{i-n+1,j-m+1} - H_{i,j} & H_{i-n+1,j-m+2} - H_{i,j} & \cdots & H_{i-n+1,j} - H_{i,j} \\ H_{i-n+2,j-m+1} - H_{i,j} & H_{i-n+2,j-m+2} - H_{i,j} & \cdots & H_{i-n+2,j} - H_{i,j} \\ \vdots & \vdots & & \vdots \\ H_{i,j-m+1} - H_{i,j} & H_{i,j-m+2} - H_{i,j} & \cdots & H_{i,j} - H_{i,j} \end{array}.$$

Since $\alpha > 0$ and $\beta > 0$ the sequences in rows and columns are decreasing sequences and therefore this is a plane partition.

5.2 Binary words and reconstruction of the plane partition.

We easily associate to a plane partition $PP(n,m;i,j)$ a binary $n \times m$ word by way of the transformation: $B(n,m;i,j) = PP(n,m;i,j) \bmod 2$. Conversely, given a sequence U, we would like to associate to each word $w \in L(U,n,m)$ a unique plane partition.

By construction of the discrete plane, we have

$$H_{i,j} = \lceil -i\alpha - j\beta + \gamma \rceil.$$

Thus, $H_{i,j} - H_{i,j+1} = \lceil -i\alpha - j\beta + \gamma \rceil - \lceil -i\alpha - (j+1)\beta + \gamma \rceil$. Then the quantity $H_{i,j} - H_{i,j+1}$ takes only two values $\lceil \beta \rceil$ and $\lceil \beta \rceil - 1$. We note the odd value O_h and the even value E_h. In columns, we have $H_{i,j} - H_{i+1,j} = \lceil -i\alpha - j\beta + \gamma \rceil - \lceil -(i+1)\alpha - j\beta + \gamma \rceil$. Then the quantity $H_{i,j} - H_{i+1,j}$ takes only two values $\lceil \alpha \rceil$ and $\lceil \alpha \rceil - 1$. We note the odd value O_v and the even value E_v.

Take $w \in L(U, n, m)$. Recall that w is an array

$$\begin{array}{ccc} w_{n,1} & \dots & w_{n,m} \\ \vdots & & \vdots \\ w_{1,1} & \dots & w_{1,m} \end{array}.$$

The complement of a binary word w is $\overline{w} = w + 1 \bmod 2$. In other words, $\overline{w}$ is the word w where 1 is replaced by 0 and 0 is replaced by 1. In what follows, if $w_{1,m} = 1$ then we consider the word $\overline{w}$, otherwise we work on the word w. Put $H'_{0,0} = 0$. In rows, if $w_{i,j} = w_{i,j+1}$ then $H'_{i,j} = H'_{i,j+1} + E_v$ else $H'_{i,j} = H'_{i,j+1} + O_v$. In columns, if $w_{i,j} = w_{i+1,j}$ then $H'_{i,j} = H'_{i+1,j} + E_h$ else $H'_{i,j} = H'_{i+1,j} + O_h$. We can therefore construct the plane partition PP with height $H'_{i,j}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ associated to the binary word $w \in L(U, n, m)$.

Notice that we associate to the words w and $\overline{w}$ the same plane partition. Consider a plane partition PP appearing in the discrete plane; this means that there exists $(i, j) \in \mathbb{Z}^2$ such that $PP(n, m; i, j) = PP$. Thus $H_{i,j}$ can take either an odd or an even value. We show in proposition 5.1 that if a plane partition appears in an odd position, then it also appears in an even position, and conversely.

5.3 Language.

Consider the language of the sequence $U = (U_{i,j})_{(i,j)\in\mathbb{Z}^2}$ defined by

$$U_{i,j} = H_{i,j} \bmod 2, \ \forall (i,j) \in \mathbb{Z}^2.$$

We have the following proposition.

Theorem 5.1 *If the rectangular word x is an element of $L(U, n, m)$ then the word $\overline{x}$ is also an element of $L(U, n, m)$.*

Proof. The word x is a factor of the sequence U. This means that there exist i and j such that $x = B(n, m; i, j)$. Then the interval $I(x) = \bigcap_{1 \leq i \leq n, 1 \leq j \leq m} R_{\alpha'}^{i-1} R_{\beta'}^{j-1} I_{x_{i,j}}$ is non-empty. The extremal points of the intervals are given by the points $\{i\alpha' + j\beta'\}$ with $0 \leq i \leq n-1, 0 \leq j \leq m-1$ and $\{\frac{1}{2} + i\alpha' + j\beta'\}$ with $0 \leq i \leq n-1, 0 \leq j \leq m-1$. The translation on the torus by $\frac{1}{2}$ translates extremal points to extremal points. Thus the interval $I(w) + \frac{1}{2}$ is non empty and $I(x) + \frac{1}{2} = \bigcap_{1 \leq i \leq n, 1 \leq j \leq m} R_{\alpha'}^{i-1} R_{\beta'}^{j-1} (I_{w_{i,j}} + \frac{1}{2})$. The translation by $\frac{1}{2}$ is a geometrical realization of the complement transformation. Thus by a translation of $\frac{1}{2}$, we have $I(x) + \frac{1}{2}$ which corresponds to $\overline{x}$. By density of the irrational rotation on unit circle, we have the result. $\diamondsuit$

Corollary 5.2 *If a plane partition $PP(n, m; i, j)$ appears in some fixed discrete plane with $H_{i,j}$ odd (respectively even) then there exist i', j' such that the plane partition $PP(n, m; i, j) = PP(n, m; i', j')$ and $H_{i',j'}$ is even (respectively odd).*

Proof. Suppose that the plane partition $PP(n, m; i, j)$ appears in the discrete plane with $H_{i,j}$ odd. We can construct the binary word $B(n, m; i, j)$. By proposition 5.1 the binary word $\overline{B(n, m; i, j)}$ appears in the discrete plane. This means there exist i', j' integers such that $B(n, m; i', j') = \overline{B(n, m; i, j)}$. As the method of reconstruction associates the same plane partition to x and $\overline{x}$, then $PP(n, m; i', j') = PP(n, m; i, j)$ and $H_{i',j'}$ is even. $\diamondsuit$

Theorem 5.3 *The number of $n \times m$ plane partition is $C(n, m) = nm$ for all n and m positive integers.*

Proof. We partition the set of binary words $n \times m$ into two classes. The class of words $w \in L(U, n, m)$ such that $w_{1,m} = 0$ and the class of $w \in L(U, n, m)$ such that $w_{1,m} = 1$. As the number of distinct $n \times m$ binary words is $2nm$ and by the previous proposition, the two classes have the same number of elements, equal to nm. Distinct plane partitions are associated to distinct binary words of the first class. Indeed, consider w and w' in $L(U, n, m)$ with $w \neq w'$ and $w_{1,m} = w'_{1,m} = 0$. By the method of reconstruction of the plane partitions, we associate to w and w' distinct plane partitions. Thus, the number of plane partitions is $C(n, m) = nm$. $\diamondsuit$

We show that the number of $n \times m$ plane partitions appearing in a discrete plane associated to a given plane $\mathcal{P}$ is nm for all n and m positive integers.

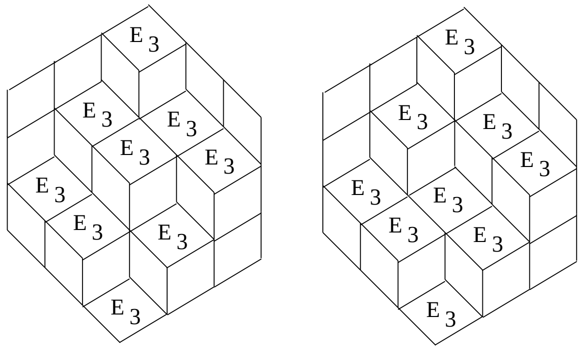


Figure 3: Two symmetric complementary patterns.

6 Symmetric complement.

We define the symmetric complement of a plane partition

$$PP(n,m;i,j) = \begin{array}{lll} H_{i-n+1,j-m+1} - H_{i,j} & \cdots & H_{i-n+1,j} - H_{i,j} \\ \vdots & & \vdots \\ H_{i,j-m+1} - H_{i,j} & \cdots & H_{i,j} - H_{i,j} \end{array} .$$

as:

$$\widetilde{PP}(n,m;i,j) \begin{array}{lll} -H_{i,j} + H_{i-n+1,j-m+1} & \cdots & -H_{i,j-m+1} + H_{i-n+1,j-m+1} \\ \vdots & & \vdots \\ -H_{i-n+1,j} + H_{i-n+1,j-m+1} & \cdots & -H_{i-n+1,j-m+1} + H_{i-n+1,j-m+1} \end{array} .$$

The symmetric complement of a plane partition $PP(n,m;i,j)$ is in geometrical terms the central symmetry by the center of the box $n \times m \times H_{i-n+1,j-m+1} - H_{i,j}$ of the three-dimensional representation of the plane partition.

Theorem 6.1 *Consider a plane partition $PP(n,m;i,j)$ appearing in a discrete plane. Then the symmetric complement $\widetilde{PP}(n,m;i,j)$ appears in the discrete plane.*

Proof. We have to show that if $PP(n,m;i,j)$ appears in the discrete plane,

$$PP(n,m;i,j) = \begin{array}{lll} H_{i-n+1,j-m+1} - H_{i,j} & \cdots & H_{i-n+1,j} - H_{i,j} \\ \vdots & & \vdots \\ H_{i,j-m+1} - H_{i,j} & \cdots & H_{i,j} - H_{i,j} \end{array} .$$

then there exists (i', j') such that

$$PP(n,m;i',j') = \begin{array}{lll} -H_{i,j} + H_{i-n+1,j-m+1} & \cdots & -H_{i,j-m+1} + H_{i-n+1,j-m+1} \\ \vdots & & \vdots \\ -H_{i-n+1,j} + H_{i-n+1,j-m+1} & \cdots & -H_{i-n+1,j-m+1} + H_{i-n+1,j-m+1} \end{array}.$$

The coding by rotation is independent of the height of the plane. That is, if we consider the plane partitions appearing in the discretization of the plane $\mathcal{P} : z = -\alpha x - \beta y + \gamma$ and the plane partitions appearing in the discretization of the plane $\mathcal{P}' : z = -\alpha x - \beta y + \gamma'$, then every plane partition appearing in the first discrete plane appears in the second. In particular, we can take γ equal to 0. If a plane partition $PP(n,m;i,j)$ appears in place (i,j), this plane partition arises from the height array:

$$\begin{array}{lll} H_{i-n+1,j-m+1} & \cdots & H_{i-n+1,j} \\ \vdots & & \vdots \\ H_{i,j-m+1} & \cdots & H_{i,j} \end{array}.$$

By central symmetry relative to the origin we have:

$$\begin{array}{lll} H_{-i,-j} & \cdots & H_{-i,-j+m-1} \\ \vdots & & \vdots \\ H_{-i+n-1,-j} & \cdots & H_{-i+n-1,-j+m-1} \end{array} = \begin{array}{lll} -H_{i,j} & \cdots & -H_{i,j-m+1} \\ \vdots & & \vdots \\ -H_{i-n+1,j} & \cdots & -H_{i-n+1,j-m+1} \end{array}.$$

Then $PP(n,m;i',j') = PP(n,m;-i+n-1,-j+m-1)$. This shows the symmetric property. $\diamondsuit$

For binary words, we define the symmetric complement of

$$w = \begin{array}{lll} w_{n,1} & \cdots & w_{n,m} \\ \vdots & & \vdots \\ w_{1,1} & \cdots & w_{1,m} \end{array}$$

as the word

$$\tilde{w} = \begin{array}{lll} w_{1,m} & \cdots & w_{1,1} \\ \vdots & & \vdots \\ w_{n,m} & \cdots & w_{n,1} \end{array}$$

Corollary 6.2 *Consider a binary word $B(n,m;i,j)$ appearing in U. Then, the symmetric complement $\tilde{B}(n,m;i,j)$ appears in U.*

Proof. This is obvious by the previous proposition. Indeed, consider the $n \times m$ binary word $B(n,m;i,j)$ associated to $PP(n,m;i,j)$ in the plane of height 0 ($z = -\alpha x - \beta y$). Thus, the binary word $B(n,m;-i+n-1,-j+m-1)$ is either $\overline{\tilde{B}(n,m;i,j)}$ (if $H_{i,j} - H_{i-n+1,j-m+1}$ is odd) or $\tilde{B}(n,m;i,j)$ (if $H_{i,j} - H_{i-n+1,j-m+1}$ is even). As x and $\overline{x}$ both appear in $L(U,n,m)$, we have the result. $\diamondsuit$

We have a new necessary condition for discrete plane. If an array of height $H = (H_{i,j})_{(i,j)\in\mathbb{Z}^2}$ remains from a discrete plane, then the associated sequences U have complexity $P(n,m) = 2nm$ for all n and m positive integers and for all x in $L(U,n,m)$, $\tilde{x}$, $\overline{x}$ and $\overline{\tilde{x}}$ are elements of $L(U,n,m)$). We think that the previous property (Complexity $P(n,m) = 2nm$ and for all x we have $\tilde{x}$, $\overline{x}$ and $\overline{\tilde{x}}$ are factors of the double sequence) is sufficient to show that the sequence U remains from a discrete plane. In other words, we guess that this property on U is a characterization of the discrete plane.

7 Generalizations.

The Theorem 5.3 is robust in all dimensions. Consider a hyperplane in $\mathbb{R}^k$ for k positive integer ($x_k = -\alpha_1 x_1 - \alpha_2 x_2 \cdots - \alpha_k x_{k-1}$) then the number of $n_1 \times n_2 \times \cdots \times n_{k-1}$ patterns corresponding to the faces of height $H_{n_1,n_2,\cdots,n_{k-1}} = \lceil -\alpha_1 n_1 - \alpha_2 n_2 \cdots - \alpha_k n_{k-1} \rceil$ is

$$C(n_1, n_2, \cdots n_{k-1}) = n_1 n_2 \cdots n_{k-1}.$$

In dimension 2, if we consider the sequence $U_{i,j} = H_{i,j} \bmod d$ for d a positive integer greater than 1, then the number of $n \times m$ modulo d words is

$$P(n,m) = dnm.$$

Consider the words modulo d in dimension k then the number of words is

$$P(n_1, n_2, \cdots n_{k-1}) = dn_1 n_2 \cdots n_{k-1}.$$

8 Acknowledgments.

I would like to thank Jean Berstel, Julien Cassaigne and Tom Schmidt for comments and conversations.

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