On the fixed points of the iterated pseudopalindromic closure

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Abstract

First introduced in the study of the Sturmian words, the iterated palindromic closure was generalized to pseudopalindromes. This operator allows one to construct words with infinitely many pseudopalindromic prefixes, called pseudostandard words. We provide here several combinatorial properties of the fixed points under the iterated pseudopalindromic closure.

Keywords: Sturmian word, palindromic closure, pseudopalindrome, pseudostandard word, fixed point, involutory antimorphism.

1 Introduction

The Sturmian words form a well-known class of infinite words over a 2-letter alphabet that occur in many different fields, for instance in astronomy, symbolic dynamics, number theory, discrete geometry, cristallography, and of course, in combinatorics on words. These words have many equivalent characterizations whose usefulness depends on the context. In discrete geometry, they are exactly the words that code the approximations of discrete lines with an irrational slope, using horizontal and diagonal moves. In symbolic dynamics, they are obtained by the exchange of 2 intervals. They are also known as the balanced aperiodic infinite words over a 2-letter alphabet. A subclass of the Sturmian words is formed by the standard Sturmian ones. For each Sturmian word, there exists a standard one having the same language, i.e., the same set of factors. A standard Sturmian word is, in a sense, the representative of all Sturmian words having the same language. All the

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words in this subclass can be obtained by a construction called the iterated palindromic closure [7]. This operation is a bijection between standard Sturmian words and non-eventually constant infinite words over a 2-letter alphabet.

On the other side, some fixed points of functions are famous in combinatorics on words. As an example, the self-generating word introduced in [17], called the Kolakoski word, is the fixed point under the run-length encoding function and raised some challenging problems. For instance, we still do not know what are its letter frequencies, if they exist. The recurrence of the Kolakoski word as well as the closure of its set of factors under complementation or reversal are other open problems.

In this context, it is a natural problem to try to characterize the fixed points under the iterated palindromic closure operator, and more generally, under the iterated pseudopalindromic closure operator, introduced in [9]. In this paper, we study these words and show some of their properties. It is organized as follows. We first give the definitions and notation used next. We recall what the iterated palindromic closure operator is and then, we introduce the iterated pseudopalindromic closure operator, which generalizes the first one using a generalization of palindromes. In Section 3, we prove the existence of fixed points under the iterated pseudopalindromic closure operator and we show them explicitly: there are 3 families of fixed points. In Section 4, we give some of their combinatorial properties, while in Section 5, we characterize the prefixes of these fixed points.

Since the words in the first family of fixed points are standard Sturmian, we use known results about standard Sturmian words in order to prove that they are not ultimately periodic, not fixed point under a nontrivial morphism and we also show some repetition properties like the greatest integer power avoided. Since the 2 others families of fixed points are not Sturmian words, proving their combinatorial properties is more difficult. We use a powerful theorem of de Luca and De Luca [9] that links a Sturmian word and an episturmian word to the fixed points of respectively the second and third families, using a morphism. Thus, we first prove properties of their associated Sturmian and episturmian words and we propagate them to the fixed points.

Notice that this paper is an extended and enhanced version of a paper presented in Salerno during the 7-th International Conference on Words [14].

2 Some iterated closures

We first recall notions on words (for more details, see for instance [18]).

An *alphabet* $\mathcal{A}$ is a finite set of symbols called *letters*. A *word over* $\mathcal{A}$ is a sequence of letters from $\mathcal{A}$. The *empty word* ε is the empty sequence. Equipped with the concatenation operation, the set $\mathcal{A}^*$ of *finite words* over $\mathcal{A}$ is a free monoid with neutral element ε and set of generators $\mathcal{A}$, and

$\mathcal{A}^+ = \mathcal{A}^* \setminus \varepsilon$. We denote by $\mathcal{A}^\omega$ the set of *(right)-infinite words* over $\mathcal{A}$. For the sake of clarity, we denote in bold character a letter denoting an infinite word, in opposition to a finite word. The set $\mathcal{A}^\infty$ is defined as the set of finite and infinite words: $\mathcal{A}^\infty = \mathcal{A}^* \cup \mathcal{A}^\omega$.

If, for some words $u, s \in \mathcal{A}^\infty$, $v, p \in \mathcal{A}^*$, $u = pvs$, then v is a *factor* of u, p is a *prefix* of u and s is a *suffix* of u. If $v \neq u$ (resp. $p \neq u$ and $s \neq u$), v is called a *proper factor* (resp. *proper prefix* and *proper suffix*). The *set of factors* of the word u is denoted $F(u)$. Two words u and v are *prefix comparable* if u is a prefix of v or v is a prefix of u. For $u = vw$, with $v \in \mathcal{A}^*$ and $w \in \mathcal{A}^\infty$, $v^{-1}u$ denotes the word w and uw^{-1} denotes the word v.

As usual, for a finite word u and a positive integer n, the *n-th power of* u, denoted u^n, is the word ε if $n = 0$; otherwise $u^n = u^{n-1}u$. A word v which is a power of a letter a is called a *block* of a's in u if $u = pvs$ with p that does not end with a nor s that does not begin with a. If $u \neq \varepsilon$, u^ω denotes the infinite word obtained by infinitely repeating u. An infinite word $\mathbf{u}$ is *periodic* (resp. *ultimately periodic*) if it can be written as $\mathbf{u} = w^\omega$ (resp. $\mathbf{u} = vw^\omega$), with $v \in \mathcal{A}^*$ and $w \in \mathcal{A}^+$. Given a finite or infinite word u, we denote by $u[i]$ the ith letter of u and by $u[i \dots j]$ the word $u[i]u[i+1] \cdots u[j]$. Given a nonempty finite word $u = u[1]u[2] \cdots u[n]$, the *length* $|u|$ of u is the integer n. One has $|\varepsilon| = 0$. The *last letter* of the word u is denoted by $\mathrm{last}(u)$. The number of occurrences of the letter a in the word u is denoted by $|u|_a$. The *frequency* of the letter a in a finite word w is $|w|_a/|w|$. For an infinite word $\mathbf{w}$, the frequency of a letter a is defined as $\lim_{n\to\infty} |\mathbf{w}[1 \dots n]|_a/n$ if it exists. If $|u|_a = 0$, then u is called *a-free word.* If for some integer $k \geq 2$ the word $u \in \mathcal{A}^\infty$ does not contain any k-th power, then u is called a *k-th power-free* word. The *rational power* of a word u is defined by $u^q = u^{\lfloor q \rfloor} p$, with $q \in \mathbb{Q}$ such that $q|u| \in \mathbb{N}$ and p is the prefix of u of length $|u|(q - \lfloor q \rfloor)$. The *critical exponent* of an infinite word $\mathbf{w}$, denoted $E(\mathbf{w})$, is the supremum of the rational powers of all its (finite) factors. There exist words such that the critical exponent is never reached. For instance, the Fibonacci word $\mathbf{f}$ has critical exponent $E(\mathbf{f}) = 2 + \phi$, where ϕ is the golden ratio, but none of the factors of $\mathbf{f}$ realize $E(\mathbf{f})$ (see [20]).

The *reversal* of the finite word $u = u[1]u[2] \cdots u[n]$, also called the *mirror image*, is $R(u)= u[n]u[n-1] \cdots u[1]$ and if $u = R(u)$, then u is called a *palindrome*. The *right-palindromic closure* (*palindromic closure*, for short) of the finite word u denoted $u^{(+)}$ is defined by $u^{(+)} = u{\cdot}R(p)$, with $u = ps$ and s is the maximal palindromic suffix of u. In other words, it is the shortest palindromic word having u as prefix.

2.1 Iterated palindromic closure

Sturmian words may be defined in many equivalent ways (see Chapter 2 in [18] for more details). For instance, they are the non-ultimately periodic infinite words over a 2-letter alphabet that have minimal complexity, that is the number of distinct factors of length n is $(n+1)$. They are also the set of non-ultimately periodic binary balanced words. Recall that a binary word w is *balanced* if for all factors f, f' having same length, and for all letters $a \in \mathcal{A}$, one has $||f|_a - |f'|_a| \leq 1$.

The Sturmian words are also the infinite non-ultimately periodic binary words that describe a discrete line. More precisely, the *Sturmian word* $\mathbf{s}_{\alpha,\rho}$, with $\alpha, \rho \in \mathbb{R}$ and $0 \leq \alpha < 1$ irrational, is the word defined as

$$\mathbf{s}[n] = \begin{cases} a \text{ if } \lfloor \alpha(n+1) + \rho \rfloor = \lfloor \alpha n + \rho \rfloor, \\ b \text{ otherwise,} \end{cases}$$

where ρ is the intercept and α the slope of the line approximated by the word $\mathbf{s}$. Recall that the slope of the word $\mathbf{s}$ is $\alpha = \lim_{n\to\infty} |s[1..n]|_b/n$.

A Sturmian word is called *standard* (or *characteristic*) if $\rho = 0$. All Sturmian words considered in this paper belong to this particular class of Sturmian words. Let us see how the iterated palindromic closure operator is hidden in the structure of the standard Sturmian words.

Given a finite word w, let us denote by $\mathrm{Pal}(w)$ the word obtained by iterating the palindromic closure:

$$\mathrm{Pal}(\varepsilon) = \varepsilon;$$
$$\mathrm{Pal}(wa) = (\mathrm{Pal}(w)a)^{(+)}, \text{ for all words } w \text{ and letters } a.$$

Note that the Pal operator is also denoted by ψ in the works of de Luca (see for instance [7]). By the definition of the iterated palindromic closure Pal, for any finite word w and letter a, $\mathrm{Pal}(w)$ is a prefix of $\mathrm{Pal}(wa)$. One can then extend the iterated palindromic closure to any infinite word $\mathbf{w} = (a[n])_{n\geq 1}$ as follows:

$$\mathrm{Pal}(\mathbf{w}) = \lim_{n\to\infty} \mathrm{Pal}(a[1]\cdots a[n]).$$

We then say that the word $\mathbf{w}$ *directs* the word $\mathrm{Pal}(\mathbf{w})$. From the works of [7], we know that Pal is a bijection between the set of binary infinite words not of the form ua^ω, with $u \in \mathcal{A}^*$ and $a \in \mathcal{A}$, and the set of standard Sturmian words. The word $\mathbf{w}$ is then called the *directive word* of the standard Sturmian word $\mathrm{Pal}(\mathbf{w})$. Note that words of the form $\mathrm{Pal}(ua^\omega)$ are periodic (see Lemma 4.1 below recalled from [10]).

The Pal operator is also well-defined over a k-letter alphabet, with $k \geq 3$. In this case, it is known [10] that $\mathrm{Pal}(\mathcal{A}^\omega)$ is exactly the set of *standard episturmian words*, a generalization over a k-letter alphabet, $k \geq 3$, of the family of standard Sturmian words (for more details, see [12]). When $\mathbf{w}$ is a word over $\mathcal{A}$ containing infinitely often each letter, then $\mathrm{Pal}(\mathbf{w})$ is called a *strict* standard episturmian word. The set of strict (standard) episturmian words corresponds to the set of (standard) Arnoux-Rauzy words [3].

Example 2.1. The *Fibonacci infinite word*

$$\mathbf{f} = \mathrm{Pal}((ab)^\omega) = \underline{a}\underline{b}a\underline{a}ba\underline{b}aaba\underline{a}babaaba\underline{b}\cdots$$

is a standard Sturmian word directed by the word $(ab)^\omega$. Indeed:

$$\begin{array}{rcl} \mathrm{Pal}(a) & = & \underline{a} \\ \mathrm{Pal}(ab) & = & (\mathrm{Pal}(a)b)^{(+)} = a\underline{b}a \\ \mathrm{Pal}(aba) & = & (\mathrm{Pal}(ab)a)^{(+)} = aba\underline{a}ba \\ \mathrm{Pal}(abab) & = & (\mathrm{Pal}(aba)b)^{(+)} = abaaba\underline{b}aaba \\ & \cdots & \end{array}$$

Example 2.2. The infinite word $abcabaac\cdots$ directs the standard episturmian word

$$\mathbf{w} = \mathrm{Pal}(abcabaac\cdots) = \underline{a}\underline{b}a\underline{c}aba\underline{a}bacaba\underline{b}acabaabacaba\underline{a}\cdots.$$

As we will do in the sequel, we have underlined in the previous examples the letters respectively in the standard Sturmian and in the standard episturmian words corresponding to the letters of their directive words, for the sake of clarity.

2.2 Iterated pseudopalindromic closure

A few years ago, de Luca and De Luca [9] have extended the notion of palindrome to what they call *pseudopalindrome*, using involutory antimorphisms. In order to define it, let us first recall that a map $\vartheta : \mathcal{A}^* \to \mathcal{A}^*$ is called an *antimorphism* of $\mathcal{A}^*$ if for all $u, v \in \mathcal{A}^*$ one has $\vartheta(uv) = \vartheta(v)\vartheta(u)$. Moreover, an antimorphism is *involutory* if $\vartheta^2 = \mathrm{id}$. A trivial involutory antimorphism is the *reversal* function R. Any involutory antimorphism ϑ of $\mathcal{A}^*$ can be constructed as $\vartheta = \tau \circ R = R \circ \tau$, with τ an involutory permutation of the alphabet $\mathcal{A}$. From now on, in order to describe an involutory antimorphism ϑ, we will then only give the involutory permutation τ of the alphabet $\mathcal{A}$. The two antimorphisms E and $\mathcal{H}$ defined respectively over $\{a, b\}$ and $\{a, b, c\}$ by

$$E = R \circ \tau \text{ with } \tau(a) = b,\ \tau(b) = a,$$
$$\mathcal{H} = R \circ \tau \text{ with } \tau(a) = a,\ \tau(b) = c,\ \tau(c) = b$$

will play, in addition to R, an important role in our study. The antimorphism E will be called, as usually, the *exchange antimorphism*. We propose to name the antimorphism $\mathcal{H}$ the *hybrid antimorphism*, hence the notation, since it contains both an identity part and an exchange part.

We can now define the generalization of palindromes given in [9]: a word $w \in \mathcal{A}^*$ is called a *ϑ-palindrome* if it is the fixed point of an involutory antimorphism ϑ of the free monoid $\mathcal{A}^*$: $\vartheta(w) = w$. When the antimorphism ϑ is not mentioned, we call it a *pseudopalindrome*.

Remark 2.3. Notice here that, when f is an antimorphism which is not involutory, it is possible that f has a finite fixed point (for instance the word acb is a fixed point of the antimorphism f defined by $f(a) = cb$, $f(b) = a$ and $f(c) = \varepsilon$). Nevertheless we observe that *a non-erasing antimorphism f has a non-empty finite fixed point w if and only if f is involutory on the alphabet of w.* Indeed if $f(w) = w$ for a non-empty word w and a non-erasing antimorphism f, then it is directly verified that $w[i] = f(w[n-i+1])$ for all $i = 1, \dots, n$.

Similarly to the palindromic closure $^{(+)}$, the *ϑ-palindromic closure* of the finite word u, also called the *pseudopalindromic closure* when the antimorphism is not specified, is defined by $u^{\oplus} = sq\vartheta(s)$, where $u = sq$, with q the longest ϑ-palindromic suffix of u. The pseudopalindromic closure of u is the shortest pseudopalindrome having u as prefix.

Example 2.4. Over the alphabet $\{a, b\}$, since the longest E-palindromic suffix of $w = aaba$ is ba, $w^{\oplus} = aaba \cdot E(aa) = aababb$.

Example 2.5. Over the alphabet $\{a, b, c\}$, let us now consider ϑ such that $\tau(a) = a$, $\tau(b) = b$ and $\tau(c) = c$, and let $w = aacbcb$. Since the longest ϑ-palindromic suffix of w is bcb, $w^{\oplus} = aacbcb \cdot \vartheta(aac) = aacbcbcaa$. Notice that in this example, $\vartheta = R \circ \tau = R \circ \mathrm{id} = R$.

Extending the Pal operator to pseudopalindrome, the Pal_{ϑ} operator is naturally defined by $\mathrm{Pal}_{\vartheta}(\varepsilon) = \varepsilon$ and $\mathrm{Pal}_{\vartheta}(wa) = (\mathrm{Pal}_{\vartheta}(w)a)^{\oplus}$, for $w \in \mathcal{A}^*$ and $a \in \mathcal{A}$. Then, for $\mathbf{w} \in \mathcal{A}^{\omega}$, $\mathrm{Pal}_{\vartheta}(\mathbf{w}) = \lim_{n\to\infty} \mathrm{Pal}_{\vartheta}(w[1] \cdots w[n])$. This limit exists since by the definition of Pal_{ϑ}, for any involutory antimorphism ϑ, $w \in \mathcal{A}^*$ and $a \in \mathcal{A}$, $\mathrm{Pal}_{\vartheta}(w)$ is a prefix of $\mathrm{Pal}_{\vartheta}(wa)$. The infinite word obtained by the Pal_{ϑ} operator is a *ϑ-standard word*, also called a *pseudostandard word* when the antimorphism is not specified. This new class of words is a general one that includes the standard Sturmian and the standard episturmian ones and was first introduced in [9].

Example 2.6. Over $\mathcal{A} = \{a, b\}$:

$$\begin{aligned}
\mathrm{Pal}_E(aab) &= (\mathrm{Pal}_E(aa)b)^{\oplus} = ((\mathrm{Pal}_E(a)a)^{\oplus}b)^{\oplus} \\
&= ((ab \cdot a)^{\oplus}b)^{\oplus} = (abab \cdot b)^{\oplus} = ababbaabab.
\end{aligned}$$

Example 2.7. Over $\mathcal{A} = \{a, b, c\}$, the $\mathcal{H}$-standard word directed by $(abc)^{\omega}$ is

$$\mathrm{Pal}_{\mathcal{H}}((abc)^{\omega}) = \underline{ab}ca\underline{c}babca\underline{a}bcacbabca\underline{b}cacbabcaabcacbabca \cdots .$$

In Section 3, we are interested in the fixed points under the Pal_{ϑ} operator: we are looking for the words $\mathbf{u} \in \mathcal{A}^{\omega}$ and the antimorphisms ϑ such that $\mathrm{Pal}_{\vartheta}(\mathbf{u}) = \mathbf{u}$. Notice that the study of the fixed points over the operator Pal_{ϑ} includes the ones over the operator Pal, since $\mathrm{Pal} = \mathrm{Pal}_R$. Indeed, one easily sees that the R-palindromes are exactly the usual ones, as we saw in Example 2.5.

3 Existence of fixed points

In this section, we prove the existence of fixed points over the iterated pseudopalindromic closure and we show which forms they can have. We denote naturally $\mathrm{Pal}_\vartheta^0(\mathbf{w}) = \mathbf{w}$ and $\mathrm{Pal}_\vartheta^n(\mathbf{w}) = \mathrm{Pal}_\vartheta(\mathrm{Pal}_\vartheta^{n-1}(\mathbf{w}))$, for any $\mathbf{w} \in \mathcal{A}^\omega$, involutory antimorphism ϑ and $n \geq 1$. Let us see some examples of the iteration of the Pal_ϑ operator over infinite words.

Examples 3.1. Over a 2-letter alphabet $\mathcal{A} = \{a, b\}$, there are only two possible involutory antimorphisms: the reversal antimorphism R and the exchange antimorphism E. Let us consider the iteration of the Pal_ϑ operator over the word $\mathbf{w} = abx\cdots$, with $x \in \mathcal{A}$.

1. Using the antimorphism R:

$$\begin{array}{rcl} \mathrm{Pal}_R(abx\cdots) & = & \underline{ab}a\underline{x}\cdots \\ \mathrm{Pal}_R^2(abx\cdots) & = & \underline{ab}a\underline{a}ba\underline{x}\cdots \\ \mathrm{Pal}_R^3(abx\cdots) & = & \underline{ab}a\underline{a}ba\underline{a}ba\underline{b}aabaaba\underline{a}babaabaaba\underline{x}\cdots\,. \end{array}$$

2. Using the antimorphism E:

$$\begin{array}{rcl} \mathrm{Pal}_E(abx\cdots) & = & \underline{a}b\underline{b}aab\underline{x}\cdots \\ \mathrm{Pal}_E^2(abx\cdots) & = & \underline{a}b\underline{b}aab\underline{b}aab\underline{a}bbaabbaab\underline{a}bbaabbaab \\ & & \underline{b}aababbaabbaababbaabbaab\underline{x}\cdots\,. \end{array}$$

In both examples, we see that the position of the letter x of the directive word $\mathbf{w}$ in $\mathrm{Pal}_R^k(\mathbf{w})$ and $\mathrm{Pal}_E^k(\mathbf{w})$ grows with the value of k. We also observe that the common prefix of $\mathrm{Pal}_\vartheta^k(\mathbf{w})$ and $\mathrm{Pal}_\vartheta^{k+1}(\mathbf{w})$ also seems to grow with k, either for $\vartheta = R$ or for $\vartheta = E$. Lemmas 3.2 and 3.3 show that only a short prefix of w is necessary to determine the word obtained by infinitely iterating the Pal_ϑ operator and yield Theorem 3.4.

Lemma 3.2. *Let $\vartheta = R \circ \tau$ be an involutory antimorphism and let $(u_k)_{k\geq 1}$ be a sequence of finite words defined by*

$$u_1 = \begin{cases} a^n b & \text{if } \tau(a) = a, \\ a & \text{if } \tau(a) = b, \end{cases}$$

and for $k \geq 2$, $u_k = \mathrm{Pal}_\vartheta(u_{k-1})$, with $a \neq b \in \mathcal{A}$, $n \geq 1$. Then $\lim_{k\to\infty} u_k$ exists.

Proof. Let us first show by induction that for all $k \geq 1$, u_k is a proper prefix of u_{k+1}. If $\tau(a) = a$, then $u_1 = a^n b$ is a proper prefix of $u_2 = \mathrm{Pal}_\vartheta(a^n b) = (\mathrm{Pal}_\vartheta(a^n)b)^\oplus = (a^n b)^\oplus \neq a^n b$. Otherwise, $\tau(a) = b$ and then, $u_1 = a$ is a proper prefix of $u_2 = \mathrm{Pal}_\vartheta(a) = ab$.

Let us suppose u_{k-1} is a proper prefix of u_k for $2 \leq k \leq n$. Then by induction and using the definition of u_n, we get

$$u_{n+1} = \mathrm{Pal}_\vartheta(u_n) = \mathrm{Pal}_\vartheta(\mathrm{Pal}_\vartheta(u_{n-1})) = \mathrm{Pal}_\vartheta(u_{n-1}w) = u_n w',$$

with $w, w' \in \mathcal{A}^+$. Thus, u_n is a proper prefix of u_{n+1}. Hence the sequence $(u_k)_{k\geq 1}$ tends to a limit. □

The limit of the sequence defined in Lemma 3.2 will be denoted by $\mathbf{s}_{\vartheta,n}$ if $\tau(a) = a$ and $\mathbf{s}_\vartheta$ otherwise.

Lemma 3.3. *Let $(u_k)_{k\geq 1}$ be the same sequence as in Lemma 3.2 and let us consider an infinite word $\mathbf{w}$ having u_1 as prefix. Then for all $k \geq 1$, u_k is a proper prefix of $\mathrm{Pal}_\vartheta^{k-1}(\mathbf{w})$.*

Proof. By the hypothesis, u_1 is a proper prefix of $\mathbf{w} = \mathrm{Pal}_\vartheta^0(\mathbf{w})$. Let us suppose that for $1 \leq k \leq n$, u_k is a proper prefix of $\mathrm{Pal}_\vartheta^{k-1}(\mathbf{w})$. Let us consider $\mathrm{Pal}_\vartheta^n(\mathbf{w})$:

$$\mathrm{Pal}_\vartheta^n(\mathbf{w}) = \mathrm{Pal}_\vartheta(\mathrm{Pal}_\vartheta^{n-1}(\mathbf{w})) = \mathrm{Pal}_\vartheta(u_n\mathbf{v}) = \mathrm{Pal}_\vartheta(u_n)\mathbf{v}' = u_{n+1}\mathbf{v}',$$

for $\mathbf{v}, \mathbf{v}' \in \mathcal{A}^\omega$. Then, u_{n+1} is a proper prefix of $\mathrm{Pal}_\vartheta^n(\mathbf{w})$. This shows that the sequence $\mathrm{Pal}_\vartheta^k(\mathbf{w})$ converges to $\mathbf{s}_\vartheta$ or $\mathbf{s}_{\vartheta,n}$, depending on τ. □

Theorem 3.4 (and definition)**.** *Over a k-letter alphabet, with $k \geq 2$, there are exactly 3 kinds of fixed points having at least 2 different letters, only depending on the first letters of the word and the involutory antimorphism $\vartheta = R \circ \tau$ considered.*

1. *When $\tau(a) = a$ and $\tau(b) = b$, with $a \neq b$, for all $n \geq 1$, Pal_ϑ has a unique fixed point beginning with $a^n b$, denoted $\mathbf{s}_{R,n,a,b}$, which equals*

$$\mathbf{s}_{R,n,a,b} = \lim_{i\to\infty} \mathrm{Pal}^i(a^n b) = \underline{a}^n \underline{b} a^n (\underline{a} b a^n)^{n+1} \underline{b} (a^{n+1} b)^{n+1} a^n \underline{a} \cdots .$$

2. *When $\tau(a) = a$ and $\tau(b) = c$ for pairwise different letters a, b, c, for all $n \geq 1$, Pal_ϑ has a unique fixed point beginning with $a^n b$, denoted by $\mathbf{s}_{\mathcal{H},n,a,b,c}$, which equals*

$$\mathbf{s}_{\mathcal{H},n,a,b,c} = \lim_{i\to\infty} \mathrm{Pal}_{\mathcal{H}}^i(a^n b) = \underline{a}^n \underline{b} c a^n \underline{c} b a^n b c a^n (\underline{a} b c a^n c b a^n b c a^n)^n \underline{c} \cdots .$$

3. *When $\tau(a) = b$ and $\tau(b) = a$, with $a \neq b$, Pal_ϑ has a unique fixed point beginning with $a^n b$ only if $n = 1$. It is denoted by $\mathbf{s}_{E,a,b}$ and equals*

$$\mathbf{s}_{E,a,b} = \lim_{i\to\infty} \mathrm{Pal}_E^i(a) = \underline{a} b \underline{b} a a b \underline{b} a a b \underline{a} b b a a b b a a b \underline{a} b b a a b b a a b \underline{b} \cdots .$$

Proof. The proof is obtained by combining Lemmas 3.2 and 3.3. □

Theorem 3.4 characterizes all possible fixed points of Pal_ϑ except the trivial fixed point of the form a^ω, which is a fixed point of Pal_ϑ using any antimorphism $\vartheta = R \circ \tau$ with $\tau(a) = a$. This trivial fixed point corresponds to the words obtained in Theorem 3.4 *1.* and *2.* with $n = \infty$.

Remark 3.5. In [9], it is mentioned that the number of involutory antimorphisms of a k-letter alphabet $\mathcal{A}$ equals the number of involutory permutations over k elements, known as

$$k! \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{1}{2^i (n-2i)! i!}.$$

Even if there exist many involutory antimorphisms for arbitrary k-letter alphabets, fixed points over the Pal_ϑ operators contain at most three letters. More precisely, the fixed points over a 3-letter alphabet $\{a, b, c\}$ starting by a can only be obtained by the antimorphism $\mathcal{H}$ (we recall that $\mathcal{H} = R \circ \tau$, with $\tau(a) = a$ and $\tau(b) = c$). Indeed, $\tau(a) = b$ yields $\mathbf{s}_{E,a,b}$ and $\tau(a) = a$ and $\tau(b) = b$ yield $\mathbf{s}_{R,n,a,b}$. Moreover, for the antimorphism E, the fixed point cannot start by a^2, since a^2 is not a prefix of $\mathrm{Pal}_E(a^2) = abab$.

Examples 3.6. For the first values of n and for the antimorphisms R and $\mathcal{H}$, we obtain the following fixed points:

$$\begin{aligned}
\mathbf{s}_{R,1,a,b} &= \underline{ab}a\underline{a}ba\underline{a}ba\underline{b}aabaaba\underline{a}babaabaaba\underline{a}babaabaaba\underline{b}aabaaba\underline{a}bab\cdots\\
\mathbf{s}_{R,2,a,b} &= \underline{aab}aa\underline{a}baa\underline{a}baa\underline{a}baa\underline{b}aaabaaabaaabaa\underline{a}baabaaabaaabaaaba\cdots\\
\mathbf{s}_{R,3,a,b} &= \underline{aaab}aaa\underline{a}baaa\underline{a}baaa\underline{a}baaa\underline{a}baaa\underline{b}aaaabaaaabaaaabaaaaba\cdots\\
\mathbf{s}_{\mathcal{H},1,a,b,c} &= \underline{ab}ca\underline{c}babca\underline{a}bcacbabca\underline{c}babcaabcacbabca\underline{b}cacbabcaabcacbabc\cdots\\
\mathbf{s}_{\mathcal{H},2,a,b,c} &= \underline{aab}caa\underline{c}baabcaa\underline{a}bcaacbaabcaa\underline{a}bcaacbaabcaa\underline{c}baabcaaabca\cdots.
\end{aligned}$$

4 Combinatorial properties of the fixed points

In this section, we consider successively the fixed points $\mathbf{s}_{R,n,a,b}$, $\mathbf{s}_{E,a,b}$ and $\mathbf{s}_{\mathcal{H},n,a,b,c}$ of the Pal_ϑ operator and we give some of their combinatorial properties. We will see that words $\mathbf{s}_{R,n,a,b}$ are Sturmian and $\mathbf{s}_{E,a,b}$ is related to a Sturmian word, whereas words $\mathbf{s}_{\mathcal{H},n,a,b,c}$ cannot be such, since they contain the three letters a, b and c. This explains why we consider the word $\mathbf{s}_{E,a,b}$ before words $\mathbf{s}_{\mathcal{H},n,a,b,c}$ contrarily to their order of introduction in Theorem 3.4.

4.1 Study of the fixed point $\mathbf{s}_{R,n,a,b}$

Here, we consider the first fixed point of the Pal_ϑ operator, with $\vartheta = R$. Thus, in what follows, instead of writing Pal_R, we will write Pal, since it is equal.

Before stating our first property, we need the following lemma.

Lemma 4.1 ([10], Theorem 3)**.** *An infinite word obtained by the* Pal *operator is ultimately periodic if and only if its directive word has the form ua^ω, with $u \in \mathcal{A}^*$ and $a \in \mathcal{A}$.*

Proposition 4.2. *For a fixed positive $n \in \mathbb{N}$, $\mathbf{s}_{R,n,a,b}$ is not ultimately periodic and consequently, is a standard Sturmian word.*

Proof. By definition of the word $\mathbf{s}_{R,n,a,b}$, $(\mathrm{Pal}^i(a^n b))_{i\geq 0}$ forms a sequence of prefixes of $\mathbf{s}_{R,n,a,b}$. The sequence of lengths of these prefixes is strictly increasing by the definition of the Pal operator. Since ba^n is a suffix of $\mathrm{Pal}^i(a^n b)$, both letters a and b occur infinitely often in $\mathbf{s}_{R,n,a,b}$. Hence $\mathbf{s}_{R,n,a,b}$ is not of the form $u\alpha^\omega$ for a word u and a letter α. Since by its definition, $\mathbf{s}_{R,n,a,b}$ equals its directive word, Lemma 4.1 implies that $\mathbf{s}_{R,n,a,b}$ is not ultimately periodic. □

Proposition 4.2 is very useful, since it allows us to use properties of standard Sturmian words in order to characterize the fixed point $\mathbf{s}_{R,n,a,b}$. Let us recall some of them.

Theorem 4.3 ([11], see also Section 2.2.2 in [18])**.** *Let $\Delta(\mathbf{w}) = a^{d_1}b^{d_2}a^{d_3}b^{d_4}\cdots$ be the directive word of an infinite standard Sturmian word $\mathbf{w}$, with $d_i \geq 1$. Then the slope of $\mathbf{w}$ has the continued fraction expansion $\alpha_{\mathbf{w}} = [0; 1 + d_1, d_2, d_3, d_4, \ldots]$.*

Theorem 4.4. [6] *The standard Sturmian word of slope α is a fixed point of some nontrivial morphism if and only if α has a continued fraction expansion of one of the following kinds:*

1. *$\alpha = [0; 1, a_0, \overline{a_1, \ldots, a_k}]$, with $a_k \geq a_0$,*
2. *$\alpha = [0; 1 + a_0, \overline{a_1, \ldots, a_k}]$, with $a_k \geq a_0 \geq 1$.*

Recall that a *morphism* (*endormorphism* if $\mathcal{A} = \mathcal{B}$) f from $\mathcal{A}^*$ to $\mathcal{B}^*$, $\mathcal{A}, \mathcal{B}$ alphabets, is a mapping from $\mathcal{A}^*$ to $\mathcal{B}^*$ such that for all words u, v over $\mathcal{A}$, $f(uv) = f(u)f(v)$. Given an endormorphism f and a letter a, if $\lim_{n\to\infty} f^n(a)$ exists, then this limit is denoted $f^\omega(a)$ and is a fixed point of f (Morphisms f^n are naturally defined by f^0 is the identity and $f^n = f^{n-1} \circ f$).

In the past, some fixed points of nontrivial morphisms, that is morphisms different from the identity, showed interesting properties, as the Fibonacci word $\mathbf{f}$ and the Thue-Morse one $\mathbf{T}$ [22] (See also for instance [2, 4, 5]). The first one is obtained by $\lim_{i\to\infty} \varphi^i(a)$, with $\varphi(a) = ab$ and $\varphi(b) = a$, while the second one is defined by $\lim_{i\to\infty} \mu^i(a)$, with $\mu(a) = ab$ and $\mu(b) = ba$. The computation of the Fibonacci word using morphisms yields

$$\varphi(a) = ab,\ \varphi^2(a) = aba,\ \varphi^3(a) = abaab,\ \varphi^4(a) = abaababa,\ \ldots,$$

$$\mathbf{f} = abaababaabaababaaba\cdots$$

as in Example 2.1, and the one of the Thue-Morse word yields

$$\mu(a) = ab,\ \mu^2(a) = abba,\ \mu^3(a) = abbabaab,\ \ldots,$$

$$\mathbf{T} = abbabaabbaababbabaababbaabbabaab\cdots.$$

A wide literature is devoted to the study of these fixed points and the known results about their generating morphism are often used in order to find some of their properties. It is thus natural to wonder if the fixed points of the Pal operator are also fixed points of some nontrivial morphisms. The answer is:

Proposition 4.5. *For a fixed n, $\mathbf{s}_{R,n,a,b}$ is not a fixed point of a nontrivial morphism.*

Proof. From Theorem 4.4, a standard Sturmian word which is a fixed point of a nontrivial morphism has a slope with an ultimately periodic continued fraction expansion. If its period has even length, then it implies, using Theorem 4.3, that its directive word is also utimately periodic and hence $\mathbf{s}_{R,n,a,b}$ itself, which is impossible by Proposition 4.2. If the period of the continued fraction expansion has odd length, then the same argument holds, using twice the period. $\square$

We denote $\alpha_{n,a,b}$ the slope associated to $\mathbf{s}_{R,n,a,b}$.

Lemma 4.6. *The continued fraction expansion of $\alpha_{n,a,b}$ has bounded partial quotients.*

Proof. One can easily see that, for the continued fraction expansion $[0; 1 + d_1, d_2, \ldots]$ of $\alpha_{n,a,b}$, each d_i belongs to $\{1, n, n+1\}$. Indeed, the 0 appears only once, as the first value of the expansion. Since $\mathbf{s}_{R,n,a,b}$ is standard Sturmian, there is one letter having only blocks of length 1 and the other letter has blocks of two consecutive lengths, n and $n+1$, with n the length of the first block prefix of $\mathbf{s}_{R,n,a,b}$. Thus, using Theorem 4.3, we get $d_1 = n$, $d_{2k} = 1$ and $d_{2k+1} \in \{n, n+1\}$ for $k \geq 1$. Since n is fixed, the conclusion follows. $\square$

Lemma 4.6 in itself is not that interesting, but we will see that it will be useful to state repetition properties of the fixed point $\mathbf{s}_{R,n,a,b}$ in Proposition 4.8. Notice that the first part of Lemma 4.7 is due to [19].

Lemma 4.7 ([23], Theorem 17)**.** *Let $\alpha > 0$ be an irrational number with $d_\alpha = [d_0; d_1, d_2, \ldots]$, its continued fraction expansion. Then the standard Sturmian word of slope α denoted $\mathbf{w}_\alpha$ is k-th power-free for some integer k if, and only if, d_α has bounded partial quotients. Moreover, if d_α has bounded partial quotients, then $\mathbf{w}_\alpha$ is k-th power-free but not $(k-1)$-th power-free for $k = 3 + \max_{i \geq 0} d_i$.*

Combining Lemmas 4.6 and 4.7, we directly get:

Proposition 4.8. *$\mathbf{s}_{R,n,a,b}$ is $(n+4)$-th power-free, but contains $(n+3)$-th powers.*

By direct computation, we easily obtain arbitrarily large prefixes of the word $\mathbf{s}_{R,n,a,b}$ for a fixed n. The continued fraction expansion of $\alpha_{R,n,a,b}$ is then obtained and yields the value of the slope. For the first values of n, we get:

$$\begin{aligned}
\alpha_{1,a,b} &= [0;2,1,2,1,2,1,1,1,2,1,2,1,2,1,1,1,2,1,2,1,2,1,1,1,2,1,2,1,1,\ldots] \\
&= 0.366095116093540422949960571470577467087211123077286\ldots \\
\alpha_{2,a,b} &= [0;3,1,3,1,3,1,3,1,2,1,3,1,3,1,3,1,3,1,2,1,3,1,3,1,3,1,3,1,2,\ldots] \\
&= 0.263762936248362388488733270234476572992585105341587\ldots \\
\alpha_{3,a,b} &= [0;4,1,4,1,4,1,4,1,4,1,3,1,4,1,4,1,4,1,4,1,4,1,3,1,4,1,4,1,4,\ldots] \\
&= 0.207106782338295017398506080110904477388983761913715\ldots \\
\alpha_{4,a,b} &= [0;5,1,5,1,5,1,5,1,5,1,5,1,4,1,5,1,5,1,5,1,5,1,5,1,5,1,4,1,5,\ldots] \\
&= 0.170820393253126826628272040633095783457508409253431\ldots \\
\alpha_{5,a,b} &= [0;6,1,6,1,6,1,6,1,6,1,6,1,6,1,5,1,6,1,6,1,6,1,6,1,6,1,6,1,6,\ldots] \\
&= 0.145497224367909729164797535715036805731190553212987\ldots
\end{aligned}$$

Whether $\alpha_{n,a,b}$ is transcendental is also an interesting problem. Using a result of Adamczewski and Bugeaud [1], we get the following answer:

Proposition 4.9. *For any $n \geq 1$, $\alpha_{n,a,b}$ is transcendental.*

Before proving this proposition, let us first recall their useful result.

Theorem 4.10. [1] *Let $\mathbf{a} = (a_\ell)_{\ell \geq 1}$ be a sequence of positive integers. If the word $\mathbf{a}$ begins in arbitrarily long palindromes, then the real number $\alpha = [0; a_1, a_2, \ldots, a_\ell, \ldots]$ is either quadratic irrational or transcendental.*

Proof of Proposition 4.9. By Proposition 4.2, $\mathbf{s}_{R,n,a,b}$ is not ultimately periodic. Consequently, there are infinitely many occurrences of a's and b's in $\mathbf{s}_{R,n,a,b}$. Let

$$P = \{i \in \mathbb{N} \setminus 0 \mid \mathbf{s}_{R,n,a,b}[i+1] = a\} \text{ and } P' = \{\mathrm{Pal}(\mathbf{s}_{R,n,a,b}[1 \ldots i]) \mid i \in P\}.$$

Both sets are infinite. Moreover, by its construction, any palindrome in the set P' is followed by an a at its first occurrence in $\mathbf{s}_{R,n,a,b}$. By Theorem 4.3 and since $\mathbf{s}_{R,n,a,b}$ equals its directive word, if $a^{i_1} b^{i_2} \cdots b^{i_2} a^{i_1}$ is a palindromic prefix of $\mathbf{s}_{R,n,a,b}$, then the continued fraction expansion of its slope begins by $[0; 1+i_1, i_2, \ldots, i_2, i_1 + \rho, \ldots]$, for some integer ρ. Moreover by Proposition 4.2, $\mathbf{s}_{R,n,a,b}$ is standard Sturmian which implies that $\rho = 0$ or $\rho = 1$ depending on the next letter occurring in $\mathbf{s}_{R,n,a,b}$. By the construction of the palindromes in P', we know that they all are palindromes such that $\rho = 1$. That implies that for any n, the continued fraction expansion of the slope of

$\mathbf{s}_{R,n,a,b}$ begins by infinitely many palindromes. We conclude using Theorem 4.10: since the continued fraction expansion of the slope is not ultimately periodic, it cannot be quadratic; hence, it is transcendental. $\square$

Notice that the previous proof works since $\mathbf{s}_{R,n,a,b}$ equals its directive word. Otherwise, the result it not necessarily true.

4.2 Study of the fixed point $\mathbf{s}_{E,a,b}$

We have seen in the previous subsection that since $\mathbf{s}_{R,n,a,b}$ are standard Sturmian words, some properties follow directly. Here, we study the fixed point $\mathbf{s}_{E,a,b}$, which equals

$$\mathbf{s}_{E,a,b} = abbaabbaababbaabbaababbaabbaab\cdots.$$

Recall that Sturmian words are known to be balanced. It is sufficient to consider the letter a and the factors bb and aa to be convinced that $\mathbf{s}_{E,a,b}$ is not balanced, and consequently, that it is not a Sturmian word.

We now recall a powerful result of de Luca and De Luca. For $\vartheta = \tau \circ R$ an involutory antimorphism over an alphabet $\mathcal{A}$, with τ an involutory permutation of $\mathcal{A}$, μ_ϑ is the morphism defined for all a in $\mathcal{A}$, by $\mu(a) = a$ if $a = \tau(a)$ and by $\mu(a) = a\tau(a)$ otherwise.

Theorem 4.11 ([9], Theorem 7.1)**.** *For any $\mathbf{w} \in \mathcal{A}^\omega$ and for any involutory antimorphism ϑ, one has*

$$\mathrm{Pal}_\vartheta(\mathbf{w}) = \mu_\vartheta(\mathrm{Pal}(\mathbf{w})).$$

Since we cannot use the known results about Sturmian words in order to prove combinatorial properties of the fixed point $\mathbf{s}_{E,a,b}$, the idea here is to first consider the word $\mathrm{Pal}(\mathbf{s}_{E,a,b})$ that will further appear to be standard Sturmian, and then to extend the properties to $\mu_E(\mathrm{Pal}(\mathbf{s}_{E,a,b}))$ which is the fixed point $\mathbf{s}_{E,a,b}$, by Theorem 4.11.

In what follows, $\mathbf{w}_E$ will denote $\mathrm{Pal}(\mathbf{s}_{E,a,b})$, that is

$$\mathbf{w}_E = \underline{a}\underline{b}a\underline{b}a\underline{a}baba\underline{a}baba\underline{b}aababaababa\underline{b}aababaababa\cdots.$$

Notice that here, μ_E is the Thue-Morse morphism, that is $\mu_E(a) = ab$ and $\mu_E(b) = ba$. Note also that $\mathbf{s}_{E,a,b} = \mu_E(\mathbf{w}_E)$, and so that $\mathbf{s}_{E,a,b} \in \{ab, ba\}^\omega$.

Proposition 4.12. *$\mathbf{w}_E$ is not ultimately periodic, and consequently, is a Sturmian word.*

Proof. By Lemma 4.1, $\mathbf{w}_E$ is ultimately periodic if and only if $\mathbf{s}_{E,a,b} = u\alpha^\omega$, for $u \in \mathcal{A}^*$ and $\alpha \in \mathcal{A}$. By its construction, $\mathbf{s}_{E,a,b}$ has infinitely many E-palindromes prefixes having the form $\mathrm{Pal}_E^i(ab) = abu_iab$, with $u_i \in \{a,b\}^*$, for i arbitrary large. Similarly to the proof of Proposition 4.2, we conclude that $\mathbf{s}_{E,a,b} \neq u\alpha^\omega$ and hence, $\mathbf{w}_E$ is not ultimately periodic: it is a standard Sturmian word. $\square$

Lemma 4.13. *Let ϑ be an involutory antimorphism over an alphabet $\mathcal{A}$. An infinite word* $\mathbf{w}$ *over $\mathcal{A}$ is ultimately periodic if and only if $\mu_\vartheta(\mathbf{w})$ is so.*

Proof. The "only if" part is immediate. Assume $\mu_\vartheta(\mathbf{w}) = uv^\omega$ for words $u \in \mathcal{A}^*$ and $v \in \mathcal{A}^+$. When v begins with a letter a such that $\mu_\vartheta(a) = a$, then a occurs in no word $\mu_\vartheta(b)$ with $b \neq a$, implying that $u = \mu_\vartheta(u')$, $v = \mu_\vartheta(v')$ for some words u', v'. Then $\mu_\vartheta(w) = \mu_\vartheta(u'v'^\omega)$. It is quite immediate that the morphism μ_ϑ is injective on infinite words (and also on finite ones). Hence $w = u'v'^\omega$ is ultimately periodic. Assume now that v begins with a letter a such that $a \neq \tau(a)$. Let $b = \tau(a)$. We have $\mu_\vartheta(a) = ab$, $\mu_\vartheta(b) = ba$ and neither a nor b occurs in $\mu_\vartheta(c)$ for $c \in \mathcal{A} \setminus \{a, b\}$. Possibly replacing v by v^2, we can assume that $|v|_a + |v|_b$ is even. Depending on the parity of $|u|_a + |u|_b$, two cases are possible: $u = \mu_\vartheta(u')$ and $v = \mu_\vartheta(v')$, or, $ua = \mu_\vartheta(u')$ and $a^{-1}va = \mu_\vartheta(v')$. Once again $\mu_\vartheta(w) = \mu_\vartheta(u'v'^\omega)$ and so $w = u'v'^\omega$ is ultimately periodic. □

Since $\mathbf{s}_{E,a,b} = \mu_E(\mathbf{w}_E)$, Proposition 4.14 is a direct corollary of Propositions 4.12 and 4.13.

Proposition 4.14. $\mathbf{s}_{E,a,b}$ *is not ultimately periodic.*

Another way to prove Proposition 4.14 is to use the following generalization of Lemma 4.1 to the Pal_ϑ operator.

Proposition 4.15. *Let ϑ be an involutory antimorphism over an alphabet $\mathcal{A}$. An infinite word obtained by the* Pal_ϑ *operator is ultimately periodic if and only if its directive word has the form $u\alpha^\omega$, with $u \in \mathcal{A}^*$ and $\alpha \in \mathcal{A}$.*

Proof. Assume $\mathbf{t} = \mathrm{Pal}_\vartheta(\mathbf{w})$ is ultimately periodic. By Theorem 4.11, $\mathbf{t} = \mu_\vartheta(\mathrm{Pal}(\mathbf{w}))$. Thus Proposition 4.15 appears as a direct corollary of Lemmas 4.1 and 4.13. □

Proposition 4.15 is interesting by itself, since it generalizes a well-known useful result of Droubay, Justin and Pirillo to pseudostandard words (see Theorem 3 in [10]).

We now consider the following corollary of Proposition 4.14.

Corollary 4.16. $\mathbf{w}_E$ *is not a fixed point of some nontrivial morphism.*

Proof. By Theorem 4.3, the continued fraction expansion of the slope of $\mathbf{w}_E$ is ultimately periodic if and only if its directive word, which is $\mathbf{s}_{E,a,b}$ by definition, is ultimately periodic. Hence by Proposition 4.14, the continued fraction expansion of the slope of $\mathbf{w}_E$ is not ultimately periodic which implies by Theorem 4.4 that $\mathbf{w}_E$ is not a fixed point of a nontrivial morphism. □

Proposition 4.17. $\mathbf{s}_{E,a,b}$ *is not a fixed point of some nontrivial morphism.*

Proof. Let us suppose by contradiction that there exists a nontrivial morphism ϕ such that $\mathbf{s}_{E,a,b} = \phi(\mathbf{s}_{E,a,b})$. Four cases can hold:

Case 1, $|\phi(a)|$ and $|\phi(b)|$ are odd. Since $\phi(a)$ is a prefix of $\mathbf{s}_{E,a,b}$ which belongs to $\{ab, ba\}^\omega$, $\phi(a) = \mu_E(u)\alpha$ for a word u and a letter $\alpha \in \{a, b\}$. Since $\phi(a)\phi(b)$ is a prefix of $\mathbf{s}_{E,a,b}$, there exists a word v such that $\phi(b) = \overline{\alpha}\mu_E(v)$ where $\overline{\alpha}$ is the complementary letter of α over the alphabet $\{a, b\}$. Since $\phi(abba) = \mu_E(u\alpha v)\overline{\alpha}\mu_E(vu)\alpha$ is a prefix of $\mathbf{s}_{E,a,b}$, there exists a word w such that $\mu_E(w) = \overline{\alpha}\mu_E(vu)\alpha$. Necessarily $w \in \overline{\alpha}^*$ and there exist integers k and ℓ such that $\phi(a) = (\alpha\overline{\alpha})^k\alpha$ and $\phi(b) = \overline{\alpha}(\alpha\overline{\alpha})^\ell$. Since abb is a prefix of $\mathbf{s}_{E,a,b}$, we have $k = \ell = 0$, and thus $\phi(a) = a$, $\phi(b) = b$ which contradicts the fact that ϕ is not the identity.

Case 2, $|\phi(a)|$ is odd and $|\phi(b)|$ is even. Since $\phi(ab)$ is a prefix of $\mathbf{s}_{E,a,b}$, we deduce that $\phi(a) = \mu_E(u)\alpha$ and $\phi(b) = \overline{\alpha}\mu_E(v)\beta$ for words u, v and letters α, β. Since $\phi(abba) = \mu_E(u\alpha v)\beta\overline{\alpha}\mu_E(v)\beta\mu_E(u)\alpha$ is a prefix of $\mathbf{s}_{E,a,b}$, $\beta = \alpha$ and $\beta\mu_E(u)\alpha = \alpha\mu_E(u)\alpha \in \{ab, ba\}^*$ which is not possible since any word in $\{ab, ba\}^*$ contains the same number of occurrences of a's as of b's.

Case 3, $|\phi(a)|$ is even and $|\phi(b)|$ is odd. Acting as previously, we deduce that $\phi(a) = \mu_E(u)$, $\phi(b) = \mu_E(v)\alpha$ for words u, v and a letter α. Moreover $\alpha\mu_E(v)\alpha$ must belong to $\{ab, ba\}^*$ which once again is impossible.

Case 4, $|\phi(a)|$ and $|\phi(b)|$ are even. In this case, $\phi = \mu_E \circ \eta$ for a morphism η. From $\mu_E(\mathbf{w}_{E,a,b}) = \mathbf{s}_{E,a,b} = \mu_E \circ \eta(\mathbf{s}_{E,a,b}) = \mu_E(\eta \circ \mu_E(\mathbf{w_E}))$ and from injectivity of μ_E over the set of infinite words, we deduce that $\mathbf{w}_{E,a,b}$ is the fixed point of $\eta \circ \mu_E$, a contradiction with Corollary 4.16.

□

Lemma 4.18. *The length of the blocks of* $\mathbf{s}_{E,a,b}$ *are* 1 *and* 2 *for both letters.*

Proof. By Theorem 4.11, $\mathbf{s}_{E,a,b} = \mu_E(\mathbf{w}_E)$. Thus, $\mathbf{s}_{E,a,b} \in \{ab, ba\}^\omega$ which implies that the maximal length of a block is 2. □

Proposition 4.19. $\mathbf{w}_E$ *contains* 4*-th powers, but is* 5*-th power-free.*

Proof. By Theorem 4.3, the partial quotients of the continued fraction expansion of the slope of $\mathbf{w}_E$ correspond to the blocks' lengths of $\mathbf{s}_{E,a,b}$, and so by Lemma 4.18 they have value 1 or 2. The statement is then a direct corollary of Lemma 4.7. □

It is known by [21] that, for all rational $q \geq 2$, a word $\mathbf{w}$ avoids repetition u^q if and only if $\mu_E(\mathbf{w})$ also avoids them. Proposition 4.19 then has the following corollary:

Corollary 4.20. $\mathbf{s}_{E,a,b}$ *contains* 4*-th powers, but is* 5*-th power-free.*

Since $\mathbf{s}_{E,a,b}$ is not a Sturmian word, it does not have a known geometrical interpretation. Thus, the notion of slope does not apply here. However, since $\mathbf{s}_{E,a,b} \in \{ab, ba\}^\omega$, we observe that the frequencies of the letters in $\mathbf{s}_{E,a,b}$ are both 1/2.

4.3 Study of the fixed point $\mathbf{s}_{\mathcal{H},n,a,b,c}$

Let us now study the properties of the last kind of fixed points. Since the words $\mathbf{s}_{\mathcal{H},n,a,b,c}$ do not have a separating letter (a letter a that precedes or follows each other letter $b \neq a$), they are not episturmian. As in the previous subsection, let $\mathbf{w}_{\mathcal{H},n}$ denote the episturmian word associated by Theorem 4.11 to the fixed point $\mathbf{s}_{\mathcal{H},n,a,b,c}$, that is:

$$\mathbf{w}_{\mathcal{H},n} = \mathrm{Pal}(\mathbf{s}_{\mathcal{H},n,a,b,c}) = \underline{a}^n \underline{b} a^n \underline{c} a^n b a^n \underline{a} b a^n c a^n b a^n \cdots .$$

As in the proofs of Propositions 4.2 and 4.12, one can see that the three letters a, b, c occur infinitely often in $\mathbf{s}_{\mathcal{H},n,a,b,c}$. Thus by Proposition 4.15 and by their construction, the $\mathbf{w}_{\mathcal{H},\mathbf{n}}$ satisfy:

Proposition 4.21. *The words* $\mathbf{w}_{\mathcal{H},n}$ *are not ultimately periodic and are strict standard episturmian words.*

Since by definition, $\mathbf{s}_{\mathcal{H},n,a,b,c} = \mu_{\mathcal{H}}(\mathbf{w}_{\mathcal{H},n})$, Lemma 4.13 implies:

Proposition 4.22. *The words* $\mathbf{s}_{\mathcal{H},n,a,b,c}$ *are not ultimately periodic.*

Let us recall a useful result from Justin and Pirillo.

Proposition 4.23. [16] *A standard strict episturmian word is a fixed point of a nontrivial morphism if and only if its directive word is periodic.*

From Propositions 4.22 and 4.23, we get:

Proposition 4.24. *The words* $\mathbf{w}_{\mathcal{H},n}$ *are not fixed points of a nontrivial morphism.*

To go further, we need to recall basic relations between so-called epistandard morphisms and the palindromic closure. For any letter a, we denote by L_a the morphism defined by $L_a(a) = a$ and $L_a(b) = ab$ when b is a letter different from a. We extend this notation to arbitrary word:

$$\begin{cases} L_\varepsilon \text{ is the identity morphism,} \\ L_{ua} = L_u \circ L_a \text{ for any word } u \text{ and letter } a. \end{cases}$$

Morphisms L_u are known to be the *pure standard episturmian morphisms*, or *pure epistandard morphisms* for short (see [16]).

The Pal operator is strongly related to epistandard morphisms by the following formula [16]:

$$\mathrm{Pal}(uv) = L_u(\mathrm{Pal}(v))\mathrm{Pal}(u), \text{ for all words } u, v \text{ and letter } a. \tag{1}$$

Also when at least two letters occur infinitely often in $\mathbf{w}$,

$$\mathrm{Pal}(\mathbf{w}) = \lim_{n\to\infty} L_{a_1\ldots a_n}(a_{n+1}). \tag{2}$$

Now we come to repetitions in $\mathbf{w}_{\mathcal{H},n}$. In [16], Justin and Pirillo provide important tools about fractional powers in episturmian words. In particular they proved:

Theorem 4.25 ([16], Theorem 5.2)**.** *Let $\mathcal{A}$ be an alphabet containing at least two different letters. Let $\mathbf{s}$ be a strict standard episturmian word over $\mathcal{A}$ directed by Δ. Assume that $\Delta = v^\omega$ (in particular Δ is periodic and $\mathbf{s}$ is the fixed point of a morphism) and let:*

- $\ell = \max\{i \mid \alpha^i$ *is a factor of* Δ *with* α *letter* $\}$,
- L *be the set of all 3-uples* (x, a, y) *such that* $xaya^\ell$ *is a prefix of* Δ, $a \in \mathcal{A}$, $|v| \leq |xay| < |v^2|$, $y \neq \varepsilon$, $a \notin \mathrm{alph}(y)$.

The critical exponent of $\mathbf{s}$ *is*

$$\ell + 2 + \sup_{(x,a,y)\in L} \left\{ \lim_{i\to\infty} \frac{|\mathrm{Pal}(v^i xa)|}{|L_{v^i xay}(a)|} \right\}.$$

With the notation of the previous result, one can observe that $L_{v^i xay}(a) = L_{v^i xay'}(ba)$, where $y = y'b$ begins with $\mathrm{Pal}(v^i xa)$, $b \neq a \in \mathcal{A}$. Indeed ba contains two different letters and Lemma 4.21 in [13] states that *for any word u containing at least two different letters and for any other word w, there exists a word u_w containing at least two different letters such that $L_w(u) = \mathrm{Pal}(w)u_w$*. Hence in the situation of the previous theorem, the critical exponent lies between $\ell + 2$ and $\ell + 3$. In particular $\mathbf{s}$ is $(\ell + 3)$-th power-free but contains an $(\ell+2)$-th power. This can be extended to a larger class of episturmian words, as follows.

Proposition 4.26. *Let $\mathbf{s}$ be a strict standard episturmian word directed by a word Δ and let ℓ denote the greatest integer i such that α^i is a factor of Δ with α a letter. Assume Δ contains at least one factor $aua^\ell va$ with a a letter and u, v non-empty words that do not contain the letter a. Then $\mathbf{s}$ is $(\ell + 3)$-th power-free but contains an $(\ell + 2)$-th power.*

Proof. Let $(v_i)_{i\geq 1}$ be the sequence of prefixes of $\mathbf{s}$ having a first letter different from the last letter (it is infinite since $\mathbf{s}$ is a strict standard episturmian word). For $i \geq 1$, denote $\mathbf{s}_i$ the standard episturmian word directed by v_i^ω.

It is straightforward that $\mathbf{s} = \lim_{i\to\infty} \mathbf{s}_i$ (since $\mathbf{s}$ and $\mathbf{s}_i$ share as prefix $\mathrm{Pal}(v_i)$ whose length grows with i). By choice of v_i, we know that

$$\max\{j \mid \alpha^j \in F(v_i^\omega), \alpha \in \mathcal{A}\} \leq \ell.$$

Hence by Theorem 4.25 each $\mathbf{s}_i$ is $(\ell+3)$-th power-free (see the discussion before the proposition). Consequently $\mathbf{s}$ is also $(\ell+3)$-th power-free.

Now by the hypotheses, $\Delta = waua^\ell va\Delta'$ with $a \in \mathcal{A}$ and $u, v \in \mathcal{A}^+$ such that $|u|_a = |v|_a = 0$. Let $\mathbf{s}'$ be the standard episturmian word directed by $va\Delta'$. The letter a occurs in $\mathbf{s}'$ and considering b the first letter of v, we see that $b \neq a$ and ab is a factor of the infinite word $\mathbf{s}'$. Since $\mathrm{Pal}(\mathbf{w}) = \lim_{n\to\infty} L_{a_1\ldots a_n}(a_{n+1})$ (by Eq. (2)), $\mathbf{s}$ contains as a factor the word $L_{waua^\ell}(ab) = L_{wau}(a^{\ell+1}b)$ and so $\mathbf{s}$ contains $L_{wa}(L_u(a)^{\ell+1}\mathrm{Pal}(u)b)$. By [16], since a does not occur in u, $L_u(a) = \mathrm{Pal}(u)a$. Consequently

$$\begin{aligned} L_{wa}(L_u(a)^{\ell+1}\mathrm{Pal}(u)b) &= L_{wa}((\mathrm{Pal}(u)a)^{\ell+1}\mathrm{Pal}(u)b) \\ &= L_w(L_a(\mathrm{Pal}(u)a)^{\ell+1}\, L_a(\mathrm{Pal}(u))ab) = L_w(L_{au}(a)^{\ell+2}b). \end{aligned}$$

Hence $\mathbf{s}$ contains the $(\ell+2)$-th powers $(L_{wau}(a))^{\ell+2}$. □

Previous proposition can be viewed as a generalization of Lemma 4.7. As a direct consequence, we have:

Corollary 4.27. *The words $\mathbf{w}_{\mathcal{H},n}$ are $(n+4)$-th power-free but contain $(n+3)$-th powers.*

We now deduce from what precedes properties of the words $\mathbf{s}_{\mathcal{H},n,a,b,c}$.

Proposition 4.28. *Let $\mathbf{s}_{\mathcal{H},n,a,b,c}$ be a fixed point of the $\mathrm{Pal}_{\mathcal{H}}$ operator, for a fixed n. Then $\mathbf{s}_{\mathcal{H},n,a,b,c}$ satisfies the following properties:*

1. *It is not an episturmian word, but is a pseudostandard word.*
2. *It is not a fixed point of some nontrivial morphism.*
3. *It is $(n+4)$-th power-free but contains $(n+3)$-th powers.*
4. *The frequencies of the letters b and c are equal.*

Proof. 1. By its construction, $\mathbf{s}_{\mathcal{H},n,a,b,c}$ is a pseudostandard word, but is not episturmian as already said at the beginning of the subsection.

2. Let ϕ be a morphism such $\mathbf{s}_{\mathcal{H},n,a,b,c} = \phi(\mathbf{s}_{\mathcal{H},n,a,b,c})$. We are going to prove that ϕ is the identity. Notice that since $\mathbf{s}_{\mathcal{H},n,a,b,c} = \mu_{\mathcal{H}}(\mathbf{w}_{\mathcal{H},n})$, the word $\mathbf{s}_{\mathcal{H},n,a,b,c}$ can be uniquely factorized over $\{a, bc, cb\}^\omega$.

   We now prove that words $\phi(a)$, $\phi(bc)$ and $\phi(cb)$ all belong to $\{a, bc, cb\}^*$. First assume that $\phi(a) = \varepsilon$. Then by Proposition 4.22, $\mathbf{s}_{\mathcal{H},n,a,b,c}$ is not ultimately periodic which implies $\phi(b) \neq \varepsilon$ and $\phi(c) \neq \varepsilon$. Then $\phi(b)$ begins with a. Moreover since $\phi(bca^nb)$ and $\phi(bca^ncba^nb)$ are factors of $\mathbf{s}_{\mathcal{H},n,a,b,c}$, we deduce that $\phi(bc)$ and $\phi(cb)$ both belong to $\{a, bc, cb\}^+$.

Assume now that $\phi(a) \neq \varepsilon$. Then $\phi(a)$ begins with a. Since $\phi(a)\phi(a)$, $\phi(a)\phi(bc)\phi(a)$ and $\phi(a)\phi(cb)\phi(a)$ are factors of $\mathbf{s}_{\mathcal{H},n,a,b,c}$, we deduce that all words $\phi(a)$, $\phi(bc)$ and $\phi(cb)$ belong to $\{a, bc, cb\}^+$.

We denote by u_a, u_{bc}, u_{cb} the words such that $\phi(a) = \mu_{\mathcal{H}}(u_a)$, $\phi(bc) = \mu_{\mathcal{H}}(u_{bc})$, $\phi(cb) = \mu_{\mathcal{H}}(u_{cb})$. We denote by η the morphism from $\{a, bc, cb\}^*$ to $\{a, b, c\}^*$ defined by $\eta(a) = u_a$, $\eta(bc)= u_{bc}$, $\eta(cb) = u_{cb}$ (this means that the value of $\eta(b)$ and $\eta(c)$ are not necessarily defined but that for all words u, v in $\{a, bc, cb\}^*$, $\eta(uv) = \eta(u)\eta(v)$). We have $\mu_{\mathcal{H}}(\mathbf{w}_{\mathcal{H},n}) = \mathbf{s}_{\mathcal{H},n,a,b,c} = \mu_{\mathcal{H}} \circ \eta(\mathbf{s}_{\mathcal{H},n,a,b,c}) = \mu_{\mathcal{H}} \circ \eta \circ \mu_{\mathcal{H}}(\mathbf{w}_{\mathcal{H},n})$. The injectivity of $\mu_{\mathcal{H}}$ over the set of infinite words implies that $\mathbf{w}_{\mathcal{H},n} = \eta \circ \mu_{\mathcal{H}}(\mathbf{w}_{\mathcal{H},n})$. By Proposition 4.24 this implies that $\eta \circ \mu_{\mathcal{H}}$ is the identity morphism over $\{a, b, c\}^*$. Thus $\eta(a) = a$, $\eta(bc) = b$, $\eta(cb) = c$ and so $\phi(a) = a$, $\phi(bc) = bc$, $\phi(cb) = cb$ which implies that ϕ is the identity morphism.

3. By Theorem 4.11, $\mathbf{s}_{\mathcal{H},n,a,b} = \mu_{\mathcal{H}}(\mathbf{w}_{\mathcal{H},n})$. Here $\mu_{\mathcal{H}}$ is defined by $\mu_{\mathcal{H}}(a) = a$, $\mu_{\mathcal{H}}(b) = bc$, $\mu_{\mathcal{H}}(c) = cb$. We let the reader verify that, for any integer $k \geq 2$, a word w (finite or infinite) contains a k-th power if and only if $\mu_{\mathcal{H}}(w)$ contains a k-th power. Then this item follows from Corollary 4.27.

4. This is once again a direct consequence of $\mathbf{s}_{\mathcal{H},n,a,b,c} = \mu_{\mathcal{H}}(\mathbf{w}_{\mathcal{H},n})$, since $\mu_{\mathcal{H}}$ is a morphism such that for every letter α, $|\mu_{\mathcal{H}}(\alpha)|_b = |\mu_{\mathcal{H}}(\alpha)|_c$.

□

5 About prefixes of fixed points

Theorem 3.4 shows that the fixed points $\mathbf{s}_{R,n,a,b}$ and $\mathbf{s}_{\mathcal{H},n,a,b,c}$ (resp. $\mathbf{s}_{E,a,b}$) of the Pal_ϑ operator are the limit of the sequence of finite words $u_1 = a^n b$ (resp. $u_1 = a$), $u_k = \mathrm{Pal}_\vartheta(u_{k-1})$ for $k \geq 2$. In particular, we can observe the following property:

> *For any prefix p of a fixed point of the* Pal_ϑ *operator, p is also a prefix of* $\mathrm{Pal}_\vartheta(p)$.

Before proving the converse (which leads to a characterization of prefixes of fixed points of the Pal_ϑ operator), let us observe a property given directly by the definition of the Pal_ϑ operator.

Fact 5.1. *For a finite word w and an involutory antimorphism $\vartheta = R \circ \tau$, the following assertions are equivalent:*

1. $\mathrm{Pal}_\vartheta(w) = w$;
2. $|\mathrm{Pal}_\vartheta(w)| = |w|$;
3. $w = a^{|w|}$ *for a letter a such that $\tau(a) = a$.*

Proof. The only difficulty concerns $2 \Rightarrow 3$ that can be proved by induction on $|w|$ using the immediate property "$|u| \leq |\mathrm{Pal}_\vartheta(u)|$ *for any word* u". $\square$

Fact 5.2. *Let w be a finite word starting by a letter a and $\vartheta = R \circ \tau$ be an involutory antimorphism such that $\tau(a) = b$, with $b \neq a$. Then $|w| < |\mathrm{Pal}_\vartheta(w)|$.*

Proof. For $w = a$, $\mathrm{Pal}_\vartheta(w) = ab$ and then, $|w| < |\mathrm{Pal}_\vartheta(w)|$. Since $\mathrm{Pal}_\vartheta(w\alpha) = (\mathrm{Pal}_\vartheta(w)\alpha)^{\oplus}$ and $|w\alpha| = |\mathrm{Pal}_\vartheta(w\alpha)|$ if and only if $|w| = |\mathrm{Pal}_\vartheta(w)|$, we conclude. $\square$

Proposition 5.3. *Let $\mathcal{A}$ be an alphabet of cardinality at least 2, and let $\vartheta = R \circ \tau$ be an involutory antimorphism over $\mathcal{A}$. A finite word w over $\mathcal{A}$ is a prefix of* $\mathrm{Pal}_\vartheta(w)$ *if and only if w is a prefix of a fixed point of* Pal_ϑ *not of the form a^ω, with $a \in \mathcal{A}$.*

Proof. As mentioned at the beginning of the section, we just have to prove the "only if" part. We act by induction on $|w|$. Case $|w| = 0$ is trivial. If $w = a^n$ for a $n \geq 1$ and a letter a, since w is a prefix of $\mathrm{Pal}_\vartheta(w)$, $\tau(a) = a$ or $n = 1$ and $\tau(a) = b$, for $a \neq b$. If $\tau(a) = a$, w is a prefix of $\mathbf{s}_{R,n,a,b}$ or $\mathbf{s}_{\mathcal{H},n,a,b,c}$ for distinct letters a, b, c, otherwise w is a prefix of $\mathbf{s}_{E,a,b}$.

Assume now that $w = a^n b$ with $a \neq b$ and $n \geq 1$. If $\tau(a) = a$ and $\tau(b) = b$, then w is a prefix of $\mathbf{s}_{R,n,a,b}$. If $\tau(a) = a$ and $\tau(b) \neq b$, then w is a prefix of $\mathbf{s}_{\mathcal{H},n,a,b,\tau(b)}$. If $\tau(a) \neq a$, then since w is a prefix of $\mathrm{Pal}_\vartheta(w)$, $n = 1$ and w is a prefix of $\mathbf{s}_{E,a,b}$.

The remaining case is $w = w'x$, with x a letter and w' containing at least two different letters. Since $w'x$ is a prefix of $\mathrm{Pal}_\vartheta(w'x)$, then w' is a prefix of $\mathrm{Pal}_\vartheta(w'x)$. By the definition of the Pal_ϑ operator, $\mathrm{Pal}_\vartheta(w')$ also is a prefix of $\mathrm{Pal}_\vartheta(w'x)$. From $|w'| \leq |\mathrm{Pal}_\vartheta(w')|$, we deduce that w' is a prefix of $\mathrm{Pal}_\vartheta(w')$ and so by induction w' is a prefix of a nontrivial fixed point $\mathbf{s}$ of Pal_ϑ. Facts 5.1 and 5.2 imply $|w'| < |\mathrm{Pal}_\vartheta(w')|$ and consequently, $|w'x| \leq |\mathrm{Pal}_\vartheta(w')|$. Using this last inequality and the fact that $w'x$ is a prefix of $\mathrm{Pal}_\vartheta(w'x) = (\mathrm{Pal}_\vartheta(w')x)^{(+)}$ and that $\mathrm{Pal}_\vartheta(w')$ is a prefix of $\mathbf{s}$, we conclude: $w'x$ is a prefix of $\mathbf{s}$. $\square$

We now continue exploring links between fixed points over the iterated palindromic closure Pal and their prefixes. Thus, we consider here only the first kind of fixed points, denoted $\mathbf{s}_{R,n,a,b}$. We will use epistandard morphisms (define before Theorem 4.25) and their relations (1) and (2) with the palindromic closure.

The next proposition provides a second characterization of prefixes of fixed points of Pal using morphisms.

Proposition 5.4. *For any finite word w over an alphabet of cardinality at least two, the following assertions are equivalent:*

1. *w is a prefix of a word $\mathbf{s}_{R,n,a,b}$ for different letters a, b and an integer $n \geq 1$;*

2. *there exists a letter α such that w is a prefix of $L_w(\alpha)$;*

3. *w is the power of a letter or w is a prefix of $L_w(\alpha)$ with α a letter occurring in w such that $\alpha \neq \mathrm{last}(w)$.*

Remark 5.5. In the second assertion of Proposition 5.4, one cannot replace "there exists" by "for all" as shown for instance by the word $abaa$ which is a prefix of $\mathbf{s}_{R,1,a,b}$ but not a prefix of $L_{abaa}(a) = aba$.

In order to prove Proposition 5.4, we need next lemma.

Lemma 5.6 ([15], Lemma 2.4 1)**.** *Let $w \in \mathcal{A}^*, y \in \mathcal{A}$. If w is not y-free, then we write $w = v_1 y v_2$ with v_2 y-free and the following holds:*

$$L_w(y) = \begin{cases} \mathrm{Pal}(w)y & \text{if } w \text{ is } y\text{-free;} \\ \mathrm{Pal}(w)\mathrm{Pal}(v_1)^{-1} & \text{otherwise.} \end{cases}$$

Proposition 5.7. *Let $p \in \mathcal{A}^*$ and $a \in \mathcal{A}$. Then the following are equivalent:*

1. *$L_p(a) = \mathrm{Pal}(p)$;*

2. *$|p|_a = 1$ and $p[1] = a$.*

Proof. $1 \Rightarrow 2$.

i) If $p[1] \neq a$, then $L_p(a)[1] = p[1] \neq a$ and $\mathrm{last}(L_p(a)) = a$: contradiction, since $L_p(a)$ is palindromic.

ii) If $p[1] = a$ and $|p|_a \geq 2$, then p can be written as $p = ap_1ap_2$, with $|p_2|_a = 0$. By Lemma 5.6, we have: $L_p(a) = \mathrm{Pal}(p)\mathrm{Pal}(ap_1)^{-1}$. Since $\mathrm{Pal}(ap_1)$ is not empty, we get a contradiction: $L_p(a) \neq \mathrm{Pal}(p)$.

Thus, the only possibility is $p[1] = a$ and $|p|_a = 1$.

$2 \Rightarrow 1$. If $|p|_a = 1$ and $p[1] = a$, then $p = ap'$, with $|p'|_a = 0$. By Lemma 5.6, $L_p(a) = \mathrm{Pal}(p)\mathrm{Pal}(\varepsilon)^{-1} = \mathrm{Pal}(p)$. ☐

Proposition 5.8. *Let $p \in \mathcal{A}^*$ and $a \in \mathcal{A}$. If $L_p(a)$ is palindromic, then*

1. *either $L_p(a) = \mathrm{Pal}(p)$;*

2. *or $L_p(a) = a$.*

Proof. We have seen in the proof of Proposition 5.7 (see i)), that $L_p(a)$ palindromic implies $p[1] = a$. If $|p|_a = 1$ and $p[1] = a$, then by Proposition 5.7, $L_p(a) = \mathrm{Pal}(p)$. Assume $|p|_a \geq 2$ and $p[1] = a$. Then there exist words p_1 and p_2 such that $p = ap_1ap_2$, $|p_2|_a = 0$. By Lemma 5.6 and Eq. (1) for the second equality), we have:

$$\begin{aligned} L_p(a) = \mathrm{Pal}(p)\mathrm{Pal}(ap_1)^{-1} &= L_{ap_1}(\mathrm{Pal}(ap_2))\mathrm{Pal}(ap_1)\mathrm{Pal}(ap_1)^{-1} \\ &= L_{ap_1}(\mathrm{Pal}(ap_2)). \end{aligned} \tag{3}$$

a) If $p_2 \neq \varepsilon$. Then $\mathrm{Pal}(ap_2) = ax_1ax_2 \dots ax_ka$, $k \geq 1$, with $x_i \in \mathcal{A} \setminus \{a\}$ for $1 \leq i \leq k$, and consequently,

$$L_p(a) = L_{ap_1}(a)L_{ap_1}(x_1) \cdots L_{ap_1}(x_k)L_{ap_1}(a).$$

Since $L_p(a)$ is a palindrome, $L_{ap_1}(x_1)[1] = \mathrm{last}(L_{ap_1}(x_k))$. Moreover, $L_{ap_1}(x_1)[1] = a$ and $x_k \neq a$ implies $\mathrm{last}(L_{ap_1}(x_k)) \neq a$: contradiction.

b) If $p_2 = \varepsilon$ and p is not a power of a, then let us rewrite $p = p_{11}ap_{12}a^n$, for some $n > 0$ and words p_{11} and p_{12} such that $|p_{12}|_a = 0$ and $p_{12} \neq \varepsilon$. We have $L_p(a) = L_{p_{11}ap_{12}a^n}(a) = L_{p_{11}ap_{12}}(a)$ which is not a palindrome (by case a).

c) If $p_2 = \varepsilon$ and $p = a^n$ for some n, we easily see that $L_p(a) = a$ and $\mathrm{Pal}(p) = a^n$.

□

Proof of Proposition 5.4. $1 \Rightarrow 3$. By Proposition 5.3, if w is a prefix of a word $\mathbf{s}_{R,n,a,b}$ for different letters a, b and an integer $n \geq 1$, then w is a prefix of $\mathrm{Pal}(w)$. Assume w is not a power of a letter and let α be a letter occurring in w such that $\alpha \neq \mathrm{last}(w)$. Then one can verify by induction over the length of w that $|w| \leq |L_w(\alpha)|$. Since by Lemma 5.6, $L_w(\alpha)$ is a prefix of $\mathrm{Pal}(w)$, we deduce that w is a prefix of $L_w(\alpha)$.

$3 \Rightarrow 2$ is immediate except if w is a power of a letter. But in this case, a being the letter such that $w = a^{|w|}$ and α being any other letter, w is a prefix of $L_w(\alpha)$.

$2 \Rightarrow 1$. If α occurs in w, then by Lemma 5.6, $L_w(\alpha)$ is a prefix of $\mathrm{Pal}(w)$ and so by hypothesis, w is a prefix of $\mathrm{Pal}(w)$. When α does not occur in w, Lemma 5.6 implies $L_w(\alpha) = \mathrm{Pal}(w)\alpha$ and consequently w is a prefix of $\mathrm{Pal}(w)$. In both cases, Proposition 5.3 allows to conclude that w is a prefix of a word $\mathbf{s}_{R,n,a,b}$ for different letters a, b and an integer $n \geq 1$. □

Here is a third characterization of the prefixes of fixed points of Pal.

Proposition 5.9. *Let w be a word which is prefix comparable to a^nb where a, b are two different letters and $n \geq 1$ is an integer. The following assertions are equivalent:*

1. *w is a prefix of $\mathbf{s}_{R,n,a,b}$;*
2. *w is a prefix of $L_w(w)$;*
3. *w is a prefix of $L_w(a^nb)$;*
4. *there exist letters c and d and an integer $m \geq 1$ such that w is a prefix of $L_w(c^md)$.*

The proof needs next lemma.

Lemma 5.10. *Let a, b be two different letters and let $n \geq 1$ be an integer. For all words u, there exist letters c and d and an integer $m \geq 1$ such that* $\mathrm{Pal}(u)c^m d$ *is a prefix of* $L_u(a^n b)$.

Proof. We proceed by induction over the length of u. When $u = \varepsilon$, the result holds with $c = a$, $d = b$ and $m = n$. Assume $\mathrm{Pal}(u)c^m d$ is a prefix of $L_u(a^n b)$. Let α be a letter. The word $L_\alpha(\mathrm{Pal}(u)c^m d)$ is a prefix of $L_{\alpha u}(a^n b)$. When $\alpha = c$, $L_\alpha(\mathrm{Pal}(u)c^m d) = L_c(\mathrm{Pal}(u))cc^m d$ and so by Eq. (1), $L_\alpha(\mathrm{Pal}(u)c^m d) = \mathrm{Pal}(\alpha u)c^m d$. When $\alpha \neq c$, $L_\alpha(\mathrm{Pal}(u)c^m d)$ begins with $L_\alpha(\mathrm{Pal}(u))\alpha c\alpha = \mathrm{Pal}(\alpha u)c\alpha$. Thus the property holds for αu. ☐

Proof of Proposition 5.9.
$1 \Rightarrow 2$. When w is a prefix of $\mathbf{s}_{R,n,a,b}$, since $\mathbf{s}_{R,n,a,b}$ is a fixed point of Pal, $\mathrm{Pal}(w)$ is a prefix of $\mathbf{s}_{R,n,a,b}$. Since $|w| \leq |\mathrm{Pal}(w)|$, w is a prefix of $\mathrm{Pal}(w)$. Moreover by Eq. (1), $L_w(\mathrm{Pal}(w))$ is a prefix of $\mathrm{Pal}(ww)$, and by definition of Pal, $\mathrm{Pal}(w)$ is a prefix of $\mathrm{Pal}(ww)$. Since $|\mathrm{Pal}(w)| < |L_w(\mathrm{Pal}(w))|$, $\mathrm{Pal}(w)$ is a prefix of $L_w(\mathrm{Pal}(w))$, and so w is a prefix of $L_w(\mathrm{Pal}(w))$. Finally from w prefix of $\mathrm{Pal}(w)$, we deduce that $L_w(w)$ is a prefix of $L_w(\mathrm{Pal}(w))$. It is straightforward that $|w| \leq |L_w(w)|$ so that w is a prefix of $L_w(w)$.

$2 \Rightarrow 3$. One can easily verify that $|w| < |L_w(a^n b)|$. When $a^n b$ is a prefix of w, w and $L_w(a^n b)$ are both prefixes of $L_w(w)$, and so w is a prefix of $L_w(a^n b)$. Otherwise, since w and $a^n b$ are prefix comparable, w is a power of a, and $L_w(a^n b) = wa^n b$.

$3 \Rightarrow 4$ is immediate with $c = a$, $d = b$ and $m = n$.

$4 \Rightarrow 1$. By Lemma 5.10, $\mathrm{Pal}(w)$ is a prefix of $L_w(c^m d)$. Since $|w| \leq |\mathrm{Pal}(w)|$, w is a prefix of $\mathrm{Pal}(w)$. Hence by Proposition 5.3, w is a prefix of a fixed point of Pal. Since w and $a^n b$ are prefix comparable, we can deduce that w is a prefix of $\mathbf{s}_{R,n,a,b}$. ☐

6 Conclusion

Let us summarize three problems raised by the content of this paper.

It is easy to see that any infinite word which is k-th power-free for an integer k has a critical exponent. This is the case of all words studied in this paper. An open question is to find closed formulas of the values of the critical exponent of words $\mathbf{s}_{R,n,a,b}$, $\mathbf{s}_{\mathcal{H},n,a,n}$ and $\mathbf{s}_{E,a,b}$.

Another direction of research would be to find a geometric interpretation of the palindromic closure. It may help find more about the fixed points of the operation we considered here.

Finally since the study of the pseudostandard words which are fixed points of the Pal_ϑ operator raises numerous intriguing questions, it might be interesting to also work with the more general families of words introduced in [8]. The first one is called the *generalized pseudostandard words*, that is the pseudostandard words directed by 2 words: the usual directive word and a word describing the antimorphism to use at each iteration. The second one is the pseudostandard words with seeds, that is the words obtained by iteration of the $\oplus_\vartheta$ operator with a non-empty word, called the seed.

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Appendix: More on Theorem 4.25

We have already mentioned that Theorem 4.25 is Theorem 5.2 in [16]. Nevertheless our formulation is slightly different from the original one which is:

Theorem 6.1 ([16], Theorem 5.2)**.** *Let* $\mathbf{s}$*,* $\mathcal{A}$*-strict standard episturmian, be generated by a morphism and* q *be the period of its directive word* $\Delta = (\delta_i)_{i\geq 1}$ *(with each* δ_i *a letter). Let* $\ell \in \mathbb{N}$ *be maximal such that* $y^\ell \in F(\Delta)$ *for some letter* y*. Let* $L = \{r, 0 \leq r < q \mid \delta_{r+1} = \delta_{r+2} = \cdots = \delta_{r+\ell}\}$ *and let* $d(r) = r+q+1-P(r+q+1)$ *for* $0 \leq r < q$*. Then the critical exponent for* $\mathbf{s}$ *is*

$$\gamma = \ell + 2 + \sup_{r\in L}\{\lim_{i\to\infty}(|u_{r+iq+1-d(r)}|/|h_{r+iq}|)\}.$$

Moreover for any letter u *in* $\mathbf{s}$ *the limit above can be obtained as a rational function with rational coefficients of the frequency* α_u *of this letter.*

To understand this statement, it is useful to recall that for $n \geq 0$:

- $P(n) = \sup\{p < n \mid \delta_p = \delta_n\}$ if this integer exists, ($P(n)$ being undefined otherwise),
- $u_n = \mathrm{Pal}(\delta_1 \cdots \delta_n)$ and
- $h_n = L_{\delta_1} L_{\delta_2} \cdots L_{\delta_n}(\delta_{n+1}) = L_{\delta_1\cdots\delta_n}(\delta_{n+1})$.

Note also that in this statement, $\mathcal{A}$ must contain at least two letters to allow the definition of ℓ, and that in this statement and from now on we denote δ_i the ith letter of Δ instead of $\Delta[i]$.

From now on, we explain the equivalence between the two statements. First note that symbols Δ, $\mathbf{s}$ and ℓ denote the same objects, and $q = |v|$. By sake of clarity, by now on, we denote :

$$L_{4.25} = \{(x,a,y) \mid xaya^\ell \text{ is a prefix of } \Delta, a \in \mathcal{A}, |v| \leq |xay| < |v^2|, y \neq \varepsilon, a \notin \mathrm{alph}(y)\},$$

$$L_{6.1} = \{r, 0 \leq r < q \mid \delta_{r+1} = \delta_{r+2} = \cdots = \delta_{r+\ell}\},$$

$$b_{4.25} = \sup_{(x,a,y)\in L_{4.25}} \left\{\lim_{i\to\infty} \frac{|\mathrm{Pal}(v^i xa)|}{|L_{v^i xay}(a)|}\right\},$$

$$b_{6.1} = \sup_{r\in L_{6.1}} \left\{\lim_{i\to\infty} \frac{|u_{r+iq+1-d(r)}|}{|h_{r+iq}|}\right\}.$$

We have to show that: $b_{4.25} = b_{6.1}$.

Let $(x,a,y) \in L_{4.25}$. Taking $r = |xay| - |v|$, we get $r \in [0, q[$ and $\delta_{r+1} = \cdots = \delta_{r+\ell} = a$, that is, $x \in L_{6.1}$. Observe that by definition of r, $xay = v\delta_1 \cdots \delta_r = \delta_1 \cdots \delta_{q+r}$. Since $\delta_{q+r+1} = a$ and $a \notin \mathrm{alph}(y)$, we deduce that $P(q+r+1) = |xa|$ and $d(r) = |y| + 1$ (or equivalently $|xa| = r+q+1-d(r)$). We also have $v^{i-1}xa = \delta_1 \cdots \delta_{(i-1)q+r+q+1-d(r)}$ and so $u_{r+iq+1-d(r)} = \mathrm{Pal}(v^{i-1}xa)$. Moreover $v^{i-1}xay = \delta_1 \cdots \delta_{(i-1)q+q+r}$ and so $h_{r+iq} = L_{v^{i-1}xay}(a)$. Thus $\lim_{i\to\infty} \frac{|\mathrm{Pal}(v^i xa)|}{|L_{v^i xay}(a)|} = \lim_{i\to\infty} \frac{|u_{r+iq+1-d(r)}|}{|h_{r+iq}|}$ and consequently $b_{4.25} \leq b_{6.1}$.

Let r be an integer in $L_{6.1}$. We consider the word $\Delta[1 \ldots r+q] = \delta_1 \cdots \delta_r a^\ell \delta_{r+\ell+1} \cdots \delta_{r+q}$. Since q is a period of Δ, $\delta_{r+q+1} \cdots \delta_{r+q+\ell} = a^\ell$ and so by definition of ℓ we have $\delta_{r+q} \neq a$. Let y be the word such that $a \notin \mathrm{alph}(y)$ and ay is a suffix of $a\delta_{r+\ell+1} \cdots \delta_{r+q}$ and let x be the word such that $\delta_1 \cdots \delta_{r+q} = xay$. Observe that by construction $|v| \leq |xay| < |v^2|$ and so $(x, a, y) \in L_{4.25}$. Hence for $i \geq 1$, $u_{r+iq+1-d(r)} = \mathrm{Pal}(v^{i-1}xa)$ and $h_{r+iq} = L_{v^{i-1}xay}(a)$ showing that $\lim_{i\to\infty} \frac{|u_{r+iq+1-d(r)}|}{|h_{r+iq}|} = \lim_{i\to\infty} \frac{|\mathrm{Pal}(v^i xa)|}{|L_{v^i xay}(a)|}$: $b_{6.1} \leq b_{4.25}$. This ends the proof of the equivalence between Theorems 4.25 and 6.1.