Coding rotations on intervals

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Abstract

We show that the coding of a rotation by α on m intervals can be recoded over m Sturmian words of angle α.

1 Introduction

The coding of rotations is a tool for the construction of infinite words over a finite alphabet. Consider a rotation R_α, given by an angle α, and defined for a point x by $R_\alpha(x) = \{x + \alpha\}$ where $\{y\}$ denotes the fractional part of y. Consider next a partition of the unit circle in $m+1$ half open intervals $\{B_0, B_1, \ldots B_m\}$. For any starting point x with $0 \le x < 1$, one gets an infinite word u by $I(x)I(R_\alpha(x))I(R_\alpha^2(x))\cdots I(R_\alpha^n(x))\cdots$, where $I(y) = i$ if $y \in B_i$.

In the special case where α is irrational and the partition is $B_0 = [0, \alpha[$ and $B_1 = [\alpha, 1[$, this construction produces exactly the well-known Sturmian words. These words appear in various domains as computer sciences [2], Physics, Mathematical optimization and play a crucial role in this article. It is remarkable that Sturmian words have a combinatorial characterization. Thus, they are exactly aperiodic words with (subword) complexity $p(n) = n + 1$ where the complexity function $p : \mathbb{N} \to \mathbb{N}$ counts the number of distinct factors of length n in the infinite word u [2]. The same general construction allows also to compute Rote words with complexity $p(n) = 2n$ by using an irrational rotation and the partition $B_0 = [0, \frac{1}{2}[$ and $B_1 = [\frac{1}{2}, 1[$ (see [7]). More generally, one can obtain infinite words with complexity $p(n) = an + b$, where a and b are real, by coding of rotation [1, 3].

In addition, codings of rotation with an irrational value of α and the partition $B_0 = [0, \beta[$ and $B_1 = [\beta, 1[$ are intimately related to Sturmian words. Indeed, the first sequence is the difference term by term of two Sturmian words [6]. Didier gives a characterization of the coding of rotation with a partition of m intervals of length greater than α by using Sturmian words and cellular automata [5]. Finally, Blanchard and Kurka study the complexity of formal languages that are generated by coding of rotation [4].

The goal of this article is to show that the coding of a rotation by α on $m+1$ intervals can be recoded over $m+1$ Sturmian words of angle α by the use of an automaton. Let the partition of $[0,1[$ be given, for $j = 0 \cdots m$, by $[\beta_j, \beta_{j+1}[$ where $\beta_0 = 0 < \beta_1 < \beta_2 < \cdots < \beta_j < \cdots < \beta_{m+1} = 1$, then the kth Sturmian word is given by

$$J_k(x)J_k(R_\alpha(x))J_k(R_\alpha^2(x))\cdots J_k(R_\alpha^n(x))\cdots,$$

where $J_k(y) = 1$ if $y \in I_k = [\beta_k, \beta_k + \alpha[$ and 0 otherwise (thus the $m+1$ Sturmian words have the same language).

The automaton for the recoding is a universal automaton (i.e. it is the same for all $(m+1)$tuples of Sturmian words). It has $m+1$ states. Its edges are $(m+1)$tuples of 0 and 1. The edge labeled by the vector composed of the ith letters of each Sturmian word gives the ith letter of the coding of rotation (see Figure 1).

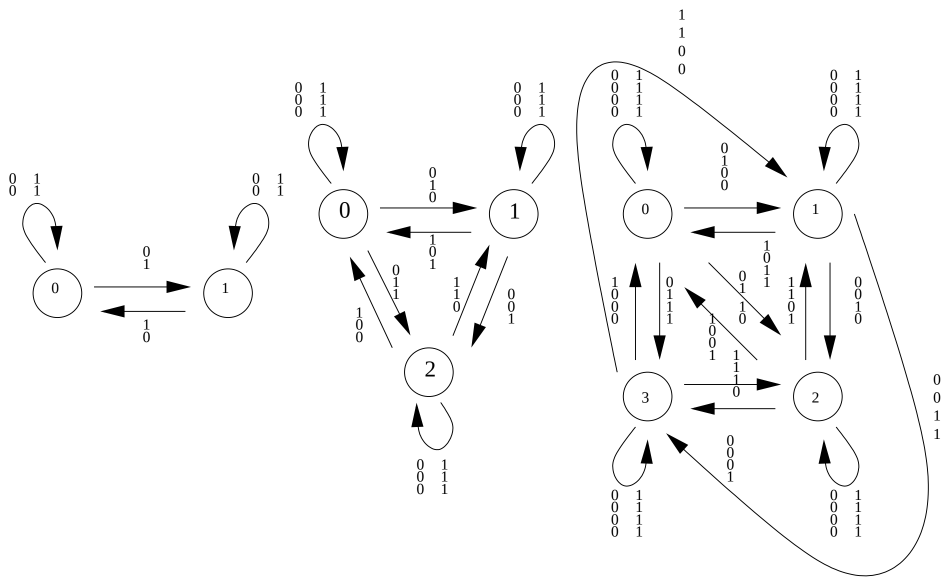

Figure 1: Automata for $m = 1, 2, 3$.

2 Examples

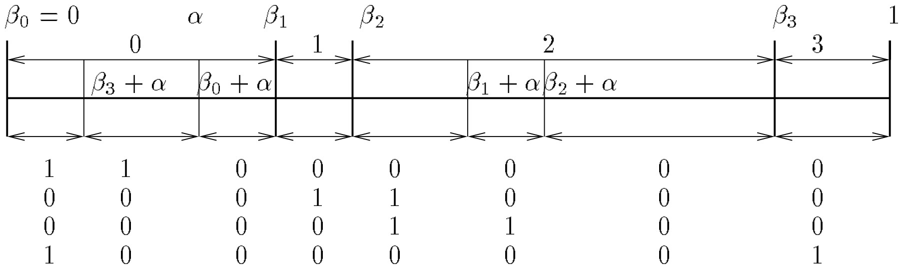

Figure 2: Partition of the unit circle.

The figure 2 shows a partition of the unit circle into 4 intervals of the form $[\beta_j, \beta_{j+1}[$. It shows also the positions of $\beta_i + \alpha$ and the coding by 8 intervals associated with binary vectors called X_K in Section 3. We can find the coding of the interval $[\beta_j, \beta_{j+1}[$ by the automaton for $m = 3$ applied to the binary vector values.

As an example, using the universal automaton for $m = 3$, the four following Sturmian words can be recoded on a word on a four letter alphabet.

$$10001001000100010\cdots$$

$$01000100100010001\cdots$$

$$00100100010001001\cdots$$

$$00010001001000100\cdots$$

is recoded on the following word:

$$01230220123012302\cdots.$$

3 Notation

We will consider subsets of $[0,1[$ that we call intervals. Let x, y be in $[0,1[$. Then we set

$$[x, y[= \begin{cases} \{z \mid x \le z < y\} & \text{if } x < y \\ \emptyset & \text{if } x = y \\ \{z \mid x \le z < 1\} \cup \{z \mid 0 \le z < y\} & \text{if } x > y \end{cases}$$

In particular, $[x,y[=[0,y[\cup[x,1[$ if $x>y$. This is precisely the notion of an interval on the torus $\mathbb{T}=\mathbb{R}/\mathbb{Z}$.

Let α, $\beta_1,\dots,\beta_m$ be numbers in the interval $]0,1[$, with $\beta_1<\cdots<\beta_m$. It will be convenient to set $\beta_0=0$ and $\beta_{m+1}=1$. The $m+1$ intervals

$$B_k=[\beta_k,\beta_{k+1}[,\qquad k=0,\dots,m$$

are a partition of $[0,1[$. We consider the *rotation* of angle α defined by $R_\alpha(x)=x+\alpha \bmod 1$. Define intervals I_k by (all values are computed modulo 1)

$$I_k=[\beta_k,\beta_k+\alpha[,\qquad k=0,\dots,m$$

We will be specially interested in the nonempty intervals

$$X_K=\bigcap_{k\in K}I_k\cap\bigcap_{k\notin K}\overline{I_k}$$

Here, K is a subset of $M=\{0,\dots,m\}$, and $\overline{I_k}=[0,1[\setminus I_k$ is the complement of I_k. Observe that, for any nonempty interval $I=[x,y[$, one gets $\overline{I}=[y,x[$.

4 Circular order

We want to compute intersections of intervals. Although the geometric approach is easy to understand, it is error prone because points are usually not in general position. Therefore, we consider a more combinatoric approach.

Given numbers $x_1,\dots,x_n\in[0,1[$, the sequence $(x_1,\dots,x_n)$ is *circularly ordered*, or *c*-ordered for short, if there exists an integer h with $1\le h\le n$ such that

$$0\le x_h\le x_{h+1}\le\cdots\le x_n\le x_1\le\cdots\le x_{h-1}<1 \tag{1}$$

If (1) holds, then either $x_1=\cdots=x_n$, or the integer h is unique. Also, if $(x_1,\dots,x_n)$ is c-ordered, then clearly $(x_2,\dots,x_n,x_1)$ is c-ordered. Any subsequence of a c-ordered sequence is c-ordered. Observe also that if $(x_1,\dots,x_n)$ is c-ordered and $x_1<x_n$ then $x_1\le\cdots\le x_n$. Indeed, if (1) holds for $h\ne1$, then $x_n\le x_1$.

Two rules are useful.

Lemma 4.1 (i) Translation Rule *If $(x_1,\dots,x_n)$ is c-ordered and $y_i\equiv x_i+\alpha \bmod 1$, then $(y_1,\dots,y_n)$ is c-ordered.*

(ii) Insertion Rule *If $(x_1,\dots,x_n)$ and $(y_1,\dots,y_m)$ are c-ordered, if furthermore $y_1\ne y_m$ and $x_i=y_1$, $x_{i+1}=y_m$, then $(x_1,\dots,x_i,y_2,\dots,y_{m-1},x_{i+1},\dots,x_n)$ is c-ordered.*

Proof. (i) We may assume $0 \leq x_1 \leq \cdots \leq x_n < 1$. The real numbers $x_i + \alpha$ satisfy $x_1 + \alpha \leq \cdots \leq x_n + \alpha < 1 + x_1 + \alpha$. If $x_n + \alpha < 1$, then $y_i = x_i + \alpha$ and $(y_1, \ldots, y_n)$ is c-ordered. Otherwise, let h be the smallest integer such that $x_h + \alpha \geq 1$. Then

$$x_1 + \alpha \leq \cdots \leq x_{h-1} + \alpha < 1 \leq x_h + \alpha \leq \cdots \leq x_n + \alpha$$

If $h = 1$, one gets $1 < x_1 + \alpha \leq \cdots \leq x_n + \alpha < 2$ and clearly $(y_1, \ldots, y_n)$ is c-ordered. If $h > 1$, then $x_n + \alpha - 1 < x_1 + \alpha$ implies

$$y_h \leq \cdots \leq y_n < y_1 \leq \cdots \leq y_{h-1}$$

(ii) There are two cases. If $x_i = \max\{x_1, \ldots, x_n\}$, then $x_{i+1} \leq \cdots \leq x_n \leq x_1 \leq \cdots \leq x_i$. From $x_i = y_1$, $x_{i+1} = y_m$, it follows that $y_m < y_1$. Let $h \neq 1$ be the integer such that $0 \leq y_h \leq \cdots \leq y_m < y_1 \leq \cdots \leq y_{h-1}$. Then

$$0 \leq y_h \leq \cdots \leq y_m = x_{i+1} \leq \cdots \leq x_n \leq x_1 \leq \cdots \leq x_i = y_1 \leq \cdots \leq y_{h-1}$$

If $x_i < \max\{x_1, \ldots, x_n\}$, then $x_i = y_1 < y_m = x_{i+1}$ and consequently $x_i = y_1 \leq y_2 \leq \cdots \leq y_m = x_{i+1}$. ∎

We observe that the insertion rule does not hold if $y_1 = y_m$. Consider the two c-ordered sequences (x, x, y) and (x, y, x), where $0 < x < y < 1$. Inserting the second into the first give the sequence (x, y, x, y) which is not c-ordered.

We prove another useful formula.

Lemma 4.2 *Let $\alpha < 1/2$. If $(x, y, x + \alpha)$ is c-ordered, then $(x, y, x + \alpha, y + \alpha)$ is c-ordered.*

Proof. The condition $\alpha < 1/2$ implies that $(x, x + \alpha, x + 2\alpha)$ is c-ordered. By the translation rule, we get that $(x + \alpha, y + \alpha, x + 2\alpha)$ is c-ordered. The insertion rule shows that $(x, x + \alpha, y + \alpha, x + 2\alpha)$ is c-ordered and, again by the insertion rule, one gets that $(x, y, x + \alpha, y + \alpha)$ is c-ordered. ∎

5 Intersection

Circular order is useful in considering intersections of intervals. Let $I = [x, y[$ be a nonempty interval. Then $x' \in [x, y[$ iff (x, x', y) is ordered. Let $I = [x, y[$ and $I' = [x', y'[$ be nonempty intervals. Then $x' \in I$ iff (x, x', y) is c-ordered. Since $I \cap I' \neq \emptyset$ iff $x' \in I$ or $x \in I'$, the intervals I and I' are disjoint iff (x, y, x') and (x', y', x) are c-ordered. Consequently, we have shown

Lemma 5.1 *Let $I = [x, y[$ and $I' = [x', y'[$ be nonempty intervals. Then $I \cap I' = \emptyset$ if and only if (x, y, x', y') is c-ordered.* ∎

The *length* s of an interval $I = [x, y[$ is the number $s = y - x$ if $x \leq y$, and is $s = 1 - (x - y)$ if $y < x$. In both cases, $y \equiv x + s \bmod 1$ so that, knowing the length, we may write $I = [x, x + s[$.

Lemma 5.2 *Let $I = [x, y[$ and $I' = [x', y'[$ be intervals of the same length $0 < \alpha < 1/2$. If I and I' intersect, then (x, x', y, y') or (x', x, y', y) is c-ordered. In the first case, $I \cap I' = [x', y[$, in the second case, $I \cap I' = [x, y'[$.*

Observe that if the length of I and I' is greater than $1/2$, then the intersection needs not to be an interval.

Proof. The discussion before Lemma 5.1 shows that I and I' intersect if and only if (x, x', y) or (x', x, y') are c-ordered. From Lemma 4.2, it follows that (x, x', y, y') or (x', x, y', y) is c-ordered. Moreover, $y \neq x'$ and $x \neq y'$ since otherwise (x, y, x', y') is c-ordered and the intervals are disjoint by Lemma 4.2. If $x = x'$ (or equivalently if $y = y'$), then $I = I'$. Thus, we may assume that the numbers x, y, x', y' are distinct.

Assume the first ordering holds. The formula for the intersection is straightforward if $0 \leq x < x' < y < y' < 1$. If $0 \leq x' < y < y' < x < 1$, then $I = [0, y[\cup [x, 1[$ and $I \cap I' = [x', y[$. The two other cases are proved in the same way. ∎

The previous lemma will be applied to the intervals $I_k = [\beta_k, \beta_k + \alpha[$. They all have same length α. We write the conclusion for further reference.

Lemma 5.3 *Let $\alpha < 1/2$. Let $I_k = [\beta_k, \beta_k + \alpha[$ and $I_\ell = [\beta_\ell, \beta_\ell + \alpha[$ be two intervals. If I_k and I_ℓ intersect then $(\beta_k, \beta_\ell, \beta_k + \alpha, \beta_\ell + \alpha)$ or $(\beta_\ell, \beta_k, \beta_\ell + \alpha, \beta_k + \alpha)$ is c-ordered. Moreover, $I_k \cap I_\ell = [\beta_\ell, \beta_k + \alpha[$ in the first case, and $I_k \cap I_\ell = [\beta_k, \beta_\ell + \alpha[$ in the second case.* ∎

The following observation is the basic step for analyzing the coding induced by a rotation. Recall that for $K \subset \{0, \ldots, m\}$,

$$X_K = \bigcap_{k \in K} I_k \cap \bigcap_{k \notin K} \overline{I_k}$$

We assume from now on that $\alpha < 1/2$.

Proposition 5.4 *Assume $X_K \neq \emptyset$ for some $K \subset \{0, \ldots, m\}$ and assume $(\beta_{i_1}, \beta_{i_2}, \beta_{i_3}, \beta_{i_4})$ is a c-ordered sequence. If $i_1, i_3 \in K$, then $i_2 \in K$ or $i_4 \in K$.*

Proof. Arguing by contradiction, suppose that $i_2, i_4 \notin K$. Since $X_K \neq \emptyset$, the interval $I_{i_1} \cap I_{i_3}$ is not empty, therefore by Lemma 5.3 $(\beta_{i_1}, \beta_{i_3}, \beta_{i_1}+\alpha, \beta_{i_3}+\alpha)$ or $(\beta_{i_3}, \beta_{i_1}, \beta_{i_3}+\alpha, \beta_{i_1}+\alpha)$ is c-ordered (or the sequence obtained by exchanging i_1 and i_3). Consider the first case, the second is the same by exchanging i_2 and i_4. Since $(\beta_{i_1}, \beta_{i_2}, \beta_{i_3})$ is c-ordered, the translation rule shows that $(\beta_{i_1}+\alpha, \beta_{i_2}+\alpha, \beta_{i_3}+\alpha)$ is c-ordered which gives, applying twice the insertion rule, that $(\beta_{i_1}, \beta_{i_2}, \beta_{i_3}, \beta_{i_1}+\alpha, \beta_{i_2}+\alpha, \beta_{i_3}+\alpha)$ is c-ordered. From this, we get that $(\beta_{i_1}, \beta_{i_2}, \beta_{i_1}+\alpha, \beta_{i_2}+\alpha)$ is c-ordered. From Lemma 5.1, we know that $I_{i_1} \cap I_{i_3} = [\beta_{i_3}, \beta_{i_1}+\alpha[$, and this is then disjoint from $\overline{I_{i_2}} = [\beta_{i_2}+\alpha, \beta_{i_2}[$. ∎

Proposition 5.5 *If X_K is not empty, then there exist integers k, ℓ with $0 \leq k < \ell \leq m$ such that $\{K, M \setminus K\} = \{\{k, \ldots, \ell-1\}, \{\ell, \ldots m, 0, \ldots, k-1\}\}$*

Proof. This is a direct consequence of the preceding discussion. ∎

It follows that there are only $(m+1)(m+2)$ intervals X_K to be considered. In fact, consider the numbers $0, \beta_1, \ldots, \beta_m, 1$ and $\alpha, \beta_1+\alpha, \ldots, \beta_m+\alpha$. They partition $[0,1[$ into exactly $2m+2$ intervals. Each of these intervals is contained in one and only one of the X_K (but $X_\emptyset$ may be scattered over several of the small intervals). This means that, among the $(m+1)(m+2)$ possible intervals X_K, there are only $2m+2$ that are used in a particular setting of the values of $\alpha, \beta_1, \ldots, \beta_m$.

Theorem 5.6 *Assume $K \neq \emptyset, M$, and $X_K \neq \emptyset$. Then $X_K = [\beta_{\ell-1}, \beta_k+\alpha[\cap[\beta_{k-1}+\alpha, \beta_\ell[$ where k and ℓ are defined in Proposition 5.5.*

If $K = \{k\}$ is a singleton, then the formula still holds with $\ell - 1 = k$.

Proof. Suppose that $K = \{k, \ldots, \ell-1\}$ with $k < \ell$. The other case is symmetric. We first prove that $\bigcap_{n \in K} I_n = [\beta_{\ell-1}, \beta_k+\alpha[$. Set $Y_K = \bigcap_{n \in K} I_n$.

Since $X_K \neq \emptyset$, the interval $I_k \cap I_{\ell-1}$ is not empty. By Lemma 4.3 there are two cases: either $(\beta_k, \beta_{\ell-1}, \beta_k+\alpha, \beta_{\ell-1}+\alpha)$ is c-ordered, or $(\beta_{\ell-1}, \beta_k, \beta_{\ell-1}+\alpha, \beta_k+\alpha)$ is c-ordered.

We show that this second case cannot happen. Indeed in this case, $Y_K \subset I_k \cap I_{\ell-1} = [\beta_k, \beta_{\ell-1}+\alpha[$. Moreover for each $n \in M \setminus K$, the sequence $(\beta_{\ell-1}, \beta_n, \beta_k)$

is c-ordered. By translation and insertion, the sequence $(\beta_{\ell-1}, \beta_n, \beta_k, \beta_{\ell-1} + \alpha, \beta_n + \alpha, \beta_k + \alpha)$ is c-ordered. This shows that $I_n \supset I_k \cap I_{\ell-1}$, and consequently $I_k \cap I_{\ell-1} \cap \overline{I_n} = \emptyset$ for each n in $M \setminus K$, contradicting the assumption that $X_K \neq \emptyset$.

Thus, $(\beta_k, \beta_{\ell-1}, \beta_k + \alpha, \beta_{\ell-1} + \alpha)$ is c-ordered. This implies that $I_k \cap I_{\ell-1} = [\beta_{\ell-1}, \beta_k + \alpha[$. If $i \in K$ then $(\beta_k, \beta_i, \beta_{\ell-1})$ is c-ordered. By translation, $(\beta_k + \alpha, \beta_i + \alpha, \beta_{\ell-1} + \alpha)$ is c-ordered. By insertion of $(\beta_k, \beta_i, \beta_{\ell-1})$ into $(\beta_k, \beta_{\ell-1}, \beta_k + \alpha, \beta_{\ell-1} + \alpha)$ one gets $(\beta_k, \beta_i, \beta_{\ell-1}, \beta_k + \alpha, \beta_{\ell-1} + \alpha)$ is c-ordered. Again by insertion of $(\beta_k + \alpha, \beta_i + \alpha, \beta_{\ell-1} + \alpha)$, the sequence $(\beta_k, \beta_i, \beta_{\ell-1}, \beta_k + \alpha, \beta_i + \alpha, \beta_{\ell-1} + \alpha)$ is c-ordered. Thus $Y_K = [\beta_{\ell-1}, \beta_k + \alpha[$.

The second part of the proof deals with $\bigcap_{n \in M \setminus K} \overline{I_n}$. In this intersection the index n runs through the set $\{0, \ldots, k-1, \ell, \ldots, m\}$. The set $M \setminus K$ is partitioned into three possibly empty subsets as follows: $n \in N$ iff $\overline{I_n} \supset Y_K$, $n \in P$ iff $\overline{I_n} \cap Y_K = [\beta_{\ell-1}, \beta_n[$ and finally $n \in Q$ iff $\overline{I_n} \cap Y_K = [\beta_n + \alpha, \beta_k + \alpha[$. Of course,

$$X_K = \bigcap_{n \in N} (\overline{I_n} \cap Y_K) \cap \bigcap_{n \in P} (\overline{I_n} \cap Y_K) \cap \bigcap_{n \in Q} (\overline{I_n} \cap Y_K)$$

If one of the sets N, P, Q is empty it does not contribute to the intersection.

Clearly $\bigcap_{n \in N} (\overline{I_n} \cap Y_K) = Y_K$. Next $\bigcap_{n \in P} (\overline{I_n} \cap Y_K) = \bigcap_{n \in P} [\beta_{\ell-1}, \beta_n[$. If P is not empty then ℓ is in P and $\bigcap_{n \in P} (\overline{I_n} \cap Y_K) = [\beta_{\ell-1}, \beta_\ell[$. Finally, $\bigcap_{n \in Q} (\overline{I_n} \cap Y_K) = \bigcap_{n \in Q} [\beta_n + \alpha, \beta_k + \alpha[$. If Q is not empty then $k-1$ is in Q and $\bigcap_{n \in Q} (\overline{I_n} \cap Y_K) = [\beta_{k-1} + \alpha, \beta_k + \alpha[$.

To finish the proof, we just have to verify that in each case, $X_K = \bigcap_{n \in N} (\overline{I_n} \cap Y_K) \cap \bigcap_{n \in P} (\overline{I_n} \cap Y_K) \cap \bigcap_{n \in Q} (\overline{I_n} \cap Y_K)$ is equal to $[\beta_{\ell-1}, \beta_k + \alpha[\cap [\beta_{k-1} + \alpha, \beta_\ell[$.

If $P \neq \emptyset$ then $\bigcap_{n \in P} (\overline{I_n} \cap Y_K) = [\beta_{\ell-1}, \beta_\ell[$ and the sequence $(\beta_{\ell-1}, \beta_\ell, \beta_k + \alpha)$ is c-ordered (case P_1). If $P = \emptyset$ then the sequence $(\beta_{\ell-1}, \beta_k + \alpha, \beta_\ell)$ is c-ordered (case P_0). If $Q \neq \emptyset$ then $\bigcap_{n \in Q} (\overline{I_n} \cap Y_K) = [\beta_{k-1} + \alpha, \beta_k + \alpha[$ and the sequence $(\beta_{\ell-1}, \beta_{k-1} + \alpha, \beta_k + \alpha)$ is c-ordered (case Q_1). If $Q = \emptyset$ then the sequence $(\beta_{\ell-1}, \beta_k + \alpha, \beta_{k-1} + \alpha)$ is c-ordered (case Q_0).

Case $(P_1 Q_1)$. If P and Q are nonempty then $X_K = [\beta_{\ell-1}, \beta_k + \alpha[\cap [\beta_{\ell-1}, \beta_\ell[\cap [\beta_{k-1} + \alpha, \beta_k + \alpha[$. As the sequences $(\beta_{\ell-1}, \beta_\ell, \beta_k + \alpha)$ and $(\beta_{\ell-1}, \beta_{k-1} + \alpha, \beta_k + \alpha)$ are c-ordered, by the insertion rule either the sequence $(\beta_{\ell-1}, \beta_\ell, \beta_{k-1} + \alpha, \beta_k + \alpha)$ or $(\beta_{\ell-1}, \beta_{k-1} + \alpha, \beta_\ell, \beta_k + \alpha)$ is c-ordered. The first case is impossible because X_K is not empty. The second case implies that $X_K = [\beta_{\ell-1}, \beta_k + \alpha[\cap [\beta_{k-1} + \alpha, \beta_\ell[$.

Case $(P_0 Q_1)$. If $P = \emptyset$ and $Q \neq \emptyset$ then $X_K = [\beta_{\ell-1}, \beta_k + \alpha[\cap [\beta_{k-1} + \alpha, \beta_k + \alpha[$ and the sequences $(\beta_{\ell-1}, \beta_k + \alpha, \beta_\ell)$, $(\beta_{\ell-1}, \beta_{k-1} + \alpha, \beta_k + \alpha)$ are c-ordered. By insertion the sequence $(\beta_{\ell-1}, \beta_{k-1}+\alpha, \beta_k+\alpha, \beta_\ell)$ is c-ordered. Thus $X_K = [\beta_{\ell-1}, \beta_k+\alpha[\cap[\beta_{k-1}+\alpha, \beta_\ell[$.

Case (P_1Q_0) is symmetric to case (P_0Q_1).

Case (P_0Q_0). If $P=\emptyset$ and $Q=\emptyset$ then $X_K = [\beta_{\ell-1}, \beta_k+\alpha[\cap[\beta_{k-1}+\alpha, \beta_\ell[$ and the sequences $(\beta_k+\alpha, \beta_{k-1}+\alpha, \beta_{\ell-1})$, $(\beta_{k-1}+\alpha, \beta_\ell, \beta_{\ell-1})$ are c-ordered. By insertion rule either the sequence $(\beta_k+\alpha, \beta_{k-1}+\alpha, \beta_\ell, \beta_{\ell-1})$, or $(\beta_k+\alpha, \beta_{k-1}+\alpha, \beta_\ell, \beta_{\ell-1})$ is c-ordered. The first case is impossible because X_K is non empty. The second case implies that $X_K = [\beta_{\ell-1}, \beta_k+\alpha[\cap[\beta_{k-1}+\alpha, \beta_\ell[$. ∎

Remark: As an additional property, the preceding proof shows that X_K is an interval and the interior of X_K does not contain any β_i or $\{\beta_i+\alpha\}$.

6 Main result

Proposition 6.1 *If $x \in B_i$ and $x+\alpha \in B_j \cap X_K$, then $j \equiv i + |K| \bmod m+1$.*

Proof If $K \neq \emptyset$ or $K \neq M$ then $X_K = [\beta_{\ell-1}, \beta_k+\alpha[\cap[\beta_{k-1}+\alpha, \beta_\ell[$. As $x \in [\beta_i, \beta_{i+1}[$, by translation rule we have $y = x+\alpha \in [\beta_i+\alpha, \beta_{i+1}+\alpha[$. Furthermore, $y \in [\beta_j, \beta_{j+1}[$. By the preceding remark and by identification, the only possibility is $j = \ell-1$ and $i = k-1$. It follows that $j = i+|K| \bmod m+1$.

If $K=\emptyset$ then $X_K = \cap_{n\in M}\overline{I_n}$. By hypothesis we have $y = x+\alpha \in B_j$. But $y \in X_K$ implies that $y \notin [\beta_j, \beta_j+\alpha[= I_j$

If $|I_j| \geq |B_j|$ then $B_j \cap X_K$ should be empty in contradiction with the hypothesis. Thus $|I_j| < |B_j|$. The interval B_j is equal to $I_j \cup [\beta_j+\alpha, \beta_{j+1}[$. That is $x+\alpha \in [\beta_j+\alpha, \beta_{j+1}[$ and $x+\alpha \in [\beta_i+\alpha, \beta_{i+1}+\alpha[$. By identification, we have $i=j$.

If $K=M$ then $X_K = \cap_{n\in M} I_n$. As $x \in I_n$ implies $x+\alpha \notin I_n$, by contraposition $x+\alpha \in X_K$ implies $x \notin I_n$ for all n. Thus $(x, \beta_n, x+\alpha)$ is c-ordered for all $n \in M$. As $x \in B_i$ and x is not in I_i the sequence $(\beta_i, \beta_i+\alpha, x, \beta_{i+1})$ is c-ordered. Thus $(x, \beta_{i+1}, \ldots, \beta_{i+m-1}, \beta_i, x+\alpha)$ is c-ordered. Consequently X_K is equal to $[\beta_i, \beta_{i+1}+\alpha[$. As $x+\alpha \in [\beta_j, \beta_{j+1}[$ by identification we find $j=i$. ∎

From this proposition, we get the following automaton $\mathcal{A}$ (Figure 1 gives the automata for $m=1,2,3$). Its set of states is the set M in bijection with the intervals B_k. The alphabet is the set of subsets of M corresponding to the nonempty intervals X_K. As already mentioned, there are $(m+1)(m+2)$ of them. The transitions or edges are given by the proposition: (i, K, j) is a transition if $j \equiv i + |K| \bmod m+1$.

Observe that the automaton is deterministic. Also, it is universal in the following sense : for a particular setting of $\alpha, \beta_1, \ldots, \beta_m$, if the β_i and the $\beta_j + \alpha$ are two by two distinct, there are only $2m+2$ of the edges that are used. Indeed they are exactly $2m+2$ intervals in the partition and between β_i and β_{i+1} the coding is uniquely determined for all i.

Moreover, the automaton has the noteworthy property that each label provided it is neither $(0, \ldots, 0)$ nor $(1, \ldots, 1)$ appears at only one edge. Consequently, both the starting state and the arrival state of such an edge is determined by is label. We denote by

$$A(v_0, \ldots, v_m)$$

the arrival state of the edge with label $(v_0, \ldots, v_m)$.

We now explain the result in terms of infinite words. Recall that $B_i = [\beta_i, \beta_{i+1}[$ and $I_i = [\beta_i, \beta_i + \alpha[$. Let u be the infinite word defined by

$$I(x) I(R_\alpha(x)) I(R_\alpha^2(x)) \cdots I(R_\alpha^n(x)) \cdots,$$

where $I(y) = i$ if $y \in B_i$. Let s_k be the kth Sturmian word defined by

$$J_k(x) J_k(R_\alpha(x)) J_k(R_\alpha^2(x)) \cdots J_k(R_\alpha^n(x)) \cdots,$$

where $J_k(y) = 1$ if $y \in I_k = [\beta_k, \beta_k + \alpha[$ and 0 otherwise.

Theorem 6.2 *The infinite word u and the $m+1$ Sturmian words s_i are related by*

$$A(s_0(n), \ldots, s_m(n)) = u(n)$$

for all $n \geq n_0$, where n_0 is the smallest integer such that $\{s_0(n_0), \ldots, s_m(n_0)\}$ contains both 0 and 1.For $n < n_0$ the value $u(n)$ is the starting state of the edge labeled $\{s_0(n_0), \ldots, s_m(n_0)\}$.

Remark that the result is written for α irrational in order to manipulate Sturmian words. The construction of the automaton is independent of the rationality of α (this case leads to infinite periodic words).

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