Palindromic language of thin discrete planes

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Abstract. We work on the Réveillès hyperplane $\mathbb{P}(v, 0, \omega)$ with *normal vector* $v \in \mathbb{R}^d$, *shift* $\mu = 0$ and *thickness* $\omega \in \mathbb{R}$. Such a hyperplane is connected as soon as ω is greater than some value $\Omega(v, 0)$, called the connecting thickness of v with null shift. In the case where v satisfies the so called *Kraaikamp and Meester criterion*, at the connecting thickness the hyperplane has very specific properties. First of all the adjacency graph of the voxels forms a tree. This tree appeared in many works both in discrete geometry and in discrete dynamical systems. In addition, it is well known that for a finite coding of length n of discrete lines, the number of palindromes in the language is exactly $n + 1$. We extend this notion of language to labeled trees and we compute the number of distinct palindromes. In fact for our voxel adjacency trees with n letters we show that the number of palindromes in the language is also $n+1$. This result establishes a first link between combinatorics on words, palindromic languages, voxel adjacency trees and connecting thickness of Réveillès hyperplanes. It also provides a better understanding of the combinatorial structure of discrete planes.

Keywords : Discrete planes, Palindromic languages, voxel adjacency trees.

1 Introduction

We work on Réveillès hyperplanes $P(v, 0, \omega)$ with *normal vector* $v \in \mathbb{R}^d$, *shift* $\mu = 0$ and *thickness* $\omega \in \mathbb{R}$. In the case where v satisfies the so called Kraaikamp and Meester criterion [KM95], at the connecting thickness the hyperplane has very specific properties. In fact the hyperplane $\mathbb{P}(v, 0, \omega)$ is also generated by the geometric palindromic closure [DV12] according to a given directive sequence Δ. If we compute finite parts of the discrete plane by using prefixes of length m of Δ, we construct a finite component called S_m. This component S_m is composed of n points in $\mathbb{R}^d$ and the adjacency graph is in fact a tree [DV12]. These trees appear in many works in discrete geometry [BDJP14,BJJP13,DV12], in discrete dynamical systems in particular for percolation problems [KM95]. They can be seen as one among many generalizations of Christoffel words. In dimension $d = 2$, we indeed generate by geometric palindromic closure all discrete lines having null shift and irrational slope. The adjacency graph in this case is a chain with $n + 1$ nodes which may be seen as a tree with two branches. Since the work of Droubay, Justin and Pirillo [DJP01], we know that the number of palindromes in a factor of length n of every Sturmian word is exactly $n+1$. This result is based on techniques of combinatorics on words including generation of Sturmian words by palindromic closure [dL97]. The proof used extensively the notion of unioccurrence of palindromes in Sturmian words given by palindromic closure. That is the first occurrence of each prefix palindrome on a Sturmian word appears exactly at each palindromic closure step [JV00]. This key point could be generalized and we prove that for hyperplanes with null shift, the generation by palindromic closure gives birth to a geometric notion of unioccurrence of bidimensional palindromes. This is a general property for discrete objects constructed by geometric palindromic closure and, in this paper, we prove that the number of palindromes in the adjacency tree with $n+1$ nodes associated with S_m has exactly $n+1$ palindromes. In other word, each node of the adjacency tree is coded by a unique palindrome. Finally, we provide examples of trees of size n which, unlike finite words, contain more than $n+1$ palindromes.

2 Words, trees and palindromes

Given a word $w \in \Sigma^\star$, we define its *language*, noted $\mathcal{L}(w)$, as the set of all its factors, $\mathcal{L}(w) = \{u \in \Sigma^\star \mid w = pus \text{ with } p, s \in \Sigma^\star\}$. The *palindromic language* of w, noted $\mathrm{Pal}(w)$, is the restriction of $\mathcal{L}(w)$ to its palindromes, $\mathrm{Pal}(w) = \{u \in \mathcal{L}(w) \mid u \text{ is a palindrome}\}$.

For example, let $w = abaaaba$, we have $\mathrm{Pal}(w) = \{\varepsilon, a, b, aa, aaa, aba, baaab, abaaaba\}$. In this example, we have

$$|\mathrm{Pal}(w)| = |w| + 1. \tag{1}$$

These words were first considered by Droubay, Justin and Pirillo who showed, in particular, that the factors of Sturmian words reach this bound [DJP01]. Be careful, this property on the maximal number of palindromes in a finite word is called by two terms in the recent literature : "rich" following a remark in [DJP01] and "full" following a definition in [BHNR04], we call it simply "maximal number of palindromes".

The above definitions are generalized to labeled trees as follow. Let $\mathcal{T}$ be a tree with labeled edges. Given an edge e its label is denoted by $\pi(e)$. A *path* in $\mathcal{T}$ is a sequence of vertices $[v_1, v_2, \ldots, v_n]$ such that, for each i from 1 to $n-1$, (v_i, v_{i+1}) is an edge of $\mathcal{T}$. A path is called *simple* if it never passes twice on the same vertex. Since, we only consider simple paths, from now on, the word *path* is used to designate simple paths. Since, in a tree, each pair of vertices is connected by a unique path, the *language* of a labeled tree, noted $\mathcal{L}(\mathcal{T})$, is defined by the set of all possible paths in this tree. More precisely, let $P_{\mathcal{T}}(u,v) = [v_1, v_2, \ldots, v_n]$ be the path from the vertex u to the vertex v, we write $\pi_{\mathcal{T}}(u,v) = \pi(v_1,v_2)\cdots\pi(v_{n-1},v_n)$ the word obtained by concatenation of the labels of all the edges along this path, we note

$$\mathcal{L}(\mathcal{T}) = \{\pi_{\mathcal{T}}(u,v) \mid u, v \text{ are vertices of } \mathcal{T}\}.$$

Consequently, we define the *palindromic language* of a labeled tree as the restriction of its language to its palindromes, $\mathrm{Pal}(\mathcal{T}) = \{w \in \mathcal{L}(\mathcal{T}) \mid w \text{ is a palindrome}\}$. Figure 1 illustrates these definitions.

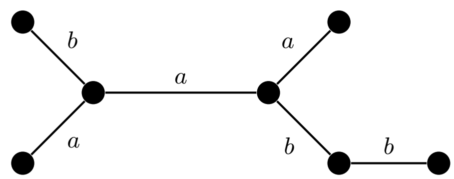

Fig. 1. A labeled tree $\mathcal{T}$ with language $\mathcal{L}(\mathcal{T}) = \{\varepsilon, a, b, aa, ab, ba, bb, aaa, aab, abb, baa, bab, bba, aabb, babb, bbaa, bbab\}$ and palindromic language $\mathrm{Pal}(\mathcal{T}) = \{\varepsilon, a, b, aa, bb, aaa, bab\}$.

Note that a word is a special case of degenerated labeled tree where all vertices form a single branch. Let $\mathrm{vert}(\mathcal{T})$ be the set of vertices of $\mathcal{T}$. Since the number of vertices in a tree is exactly one more than the number of edges, for words that satisfy Equation (1), this property is translated to trees as

$$|\mathrm{Pal}(\mathcal{T})| = |\mathrm{vert}(\mathcal{T})|. \tag{2}$$

3 Discrete hyperplanes with a tree structure

For $d \geq 2$, let $\mathcal{D} = \{1, 2, \ldots, d\}$. In the following, the canonical basis of $\mathbb{R}^d$ is $(e_i)_{i \in \mathcal{D}}$ and $\langle\, .\, ,\, .\, \rangle$ denotes the usual scalar product on $\mathbb{R}^d$. We consider arithmetical discrete hyperplanes as defined in [Rév91,AAS97]. Given a non zero vector $\mathbf{v} \in \mathbb{R}^d$ and two real numbers ω and μ, the *arithmetic discrete hyperplane* with *normal vector* $\mathbf{v}$, *shift* μ and *thickness* ω is the subset of $\mathbb{Z}^d$ defined by

$$\mathbb{P}(\mathbf{v}, \mu, \omega) = \{\mathbf{x} \in \mathbb{Z}^d \mid 0 \leq \langle \mathbf{v}, \mathbf{x} \rangle + \mu < \omega\}.$$

Given two points $x, y \in \mathbb{Z}^d$, we say that x and y are *adjacent* if there exists $i \in \{1, 2, \ldots, d\}$ such that $x = y \pm e_i$. Hereafter, we consider subsets of $\mathbb{Z}^d$ as labeled graphs using points of $\mathbb{Z}^d$ as vertices and this adjacency relation as edges. The label of an edge is given by the index of the coordinate that differs from one point to the other so that the edge $(x, x \pm e_i)$ is labeled by i. Since the adjacency relation is symmetric, we consider non-oriented edges so that the edge (x, y) is the same than (y, x). On the other hand, given a path $[x_1, x_2, \ldots, x_n]$, each edge along this path is traveled in a specific direction, either e_i or $-e_i$. Let (x, y) be an edge labeled by i, we say that a path that goes from x to y following this edge makes a *positive step* if $y - x = e_i$ while it makes a *negative step* if $y - x = -e_i$.

Following the classical terminology from graph theory, a subset $U \subset \mathbb{Z}^d$ is *connected* if its graph is connected, and U is a *tree* if it is connected and acyclic.

Given a vector $\mathbf{v} \in \mathbb{R}^d \setminus \{0\}$ and $\mu \in \mathbb{R}$, the set of thicknesses ω such that $\mathbb{P}(\mathbf{v}, \mu, \omega)$ is connected is a right unbounded interval of $\mathbb{R}_+$. Its lower bound $\Omega(\mathbf{v}, \mu)$, is known as the *connecting thickness* of $\mathbf{v}$ with shift μ. It may be computed by means of the *Fully subtractive* algorithm [BB04,JT09,DJT09,DPV14].

3.1 Construction of thin discrete planes

We consider the multidimensional continued fraction algorithm *Unordered Fully Subtractive* (UFS). We give here a short description of this algorithm. A detailed description may be found in [DPV14]. For each $k \in \mathcal{D}$, let $\sigma_k \in SL(d, \mathbb{Z})$ be such that

$$\sigma_k(v_1, v_2, \ldots, v_d) = (v_1 - v_k, \ldots, v_{k-1} - v_k,\ v_k,\ v_{k+1} - v_k, \ldots, v_d - v_k)$$

Let $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$. Given a vector $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_d) \in \mathbb{R}_+^d$, the UFS algorithm defines a sequence $(\mathbf{v}^{(n)})_{n\geq 1}$ and a *directive sequence* $\Delta = \delta_1\delta_2\cdots$ with each $\delta_i \in \mathcal{D}$ as follows. Let $\mathbf{v}^{(1)} = \mathbf{v}$ and for $n \geq 1$:

- Let k be the smallest index of a minimal coordinate of $\mathbf{v}^{(n)}$, so that

$$\mathbf{v}_k^{(n)} = \min\big(\mathbf{v}_1^{(n)}, \mathbf{v}_2^{(n)}, \ldots, \mathbf{v}_d^{(n)}\big),$$

  then, $\mathbf{v}^{(n+1)} = \sigma_k \mathbf{v}^{(n)}$ and $\delta_n = k$.

Clearly, the coordinates of the vectors $\mathbf{v}^{(n)}$ might reach 0 but may never be negative. Let $\mathcal{F}_d$ be the set of vectors such that UFS never cancels a coordinate,

$$\mathcal{F}_d = \{\mathbf{v} \in \mathbb{R}_+^d \mid \mathbf{v}^{(n)} \in \mathbb{R}_+^d \text{ for all } n \geq 0\}.$$

Note that every vector in $\mathcal{F}_d$ defines a unique infinite sequence in $\mathcal{D}^\omega$ while each non ultimately constant sequence Δ in $\mathcal{D}^\omega$ is produced by an infinity of vectors from $\mathcal{F}_d$. Indeed, multiplying a vector by a non zero constant does not alter the directive sequence. But even different directions, which means non proportional vectors, may produce the same sequence, e.g. $(1, \sqrt{2}, 3)$ and $(1, \sqrt{2}, 4)$. However, a sequence Δ in which each letter $k \in \mathcal{D}$ occurs infinitely often is produced by vectors of a unique direction. This happens if and only if $\mathbf{v}$ satisfies the criterion of Kraaikamp & Meester [KM95], namely $(d-1)\,\|\mathbf{v}^{(n)}\|_\infty < \|\mathbf{v}^{(n)}\|_1$ for all $n \geq 0$, where $\|\mathbf{v}\|_\infty = \max(\mathbf{v}_1, \ldots, \mathbf{v}_d)$ and $\|\mathbf{v}\|_1 = \mathbf{v}_1 + \cdots + \mathbf{v}_d$. When $d = 2$, this criterion becomes simply $\max(\mathbf{v}_1^{(n)}, \mathbf{v}_2^{(n)}) < \mathbf{v}_1^{(n)} + \mathbf{v}_2^{(n)}$, which is satisfied by any vector in $\mathcal{F}_2$. When $d \geq 3$, Kraaikamp and Meester [KM95] have shown that the set of vectors satisfiying this criterion has a zero Lebesgue measure. As an example, the vector $\mathbf{v} = (\alpha, \alpha + \alpha^2, 1)$, where α is the real root of $x + x^2 + x^3 - 1$, satisfies the criterion.

From a directive sequence Δ, we build a sequence $(S_n)_{n\geq 0}$ of finite subsets of $\mathbb{Z}^d$ the limit of which, S_∞, is the *geometric palindromic closure* of Δ [DV12]. This construction generalizes the one proposed in [BDJP14], and was studied in [BJJP13] for the case $d = 3$. In general, S_∞ is not an arithmetic discrete hyperplane but is always embedded in $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}, 0))$. Recent work by the same authors [DPV14] shows that if each letter $k \in \mathcal{D}$ appears infinitely often in Δ, then S_∞ is exactly $\mathbb{P}(\mathbf{v}, 0, \Omega(\mathbf{v}, 0))$.

We present two constructions, one using symmetries and the other using translations. Both constructions are equivalent and are used in order to demonstrate our main result. In all the sequel, $\mathbf{v}$ is a fixed vector in $\mathcal{F}_d$ and Δ is the associated directive sequence.

3.2 Construction by symmetries

Let $\mathrm{sym}_y(x)$ denote the homothetic transformation of x with center y and scale -1, that is $\mathrm{sym}_y(x) = 2y - x$. We build a sequence $(S_n)_{n\geq 0}$ of finite subsets of $\mathbb{Z}^d$. For this purpose, we use auxiliary sequences $(X_n)_{n\geq 0}$ and, for each $i \in \mathcal{D}$, $(Y_n^{(i)})_{n\geq 0}$ and $(s_n^{(i)})_{n\geq 0}$, where $X_n, Y^{(i)} \in (\frac{1}{2}\mathbb{Z})^d$, and $s_n^{(i)} \subset \mathbb{Z}^d$.

The construction process is:

- Initialization:
  $S_0 = \{0\} \subset \mathbb{Z}^d$, $X_0 = 0 \in \mathbb{Z}^d$ and for each $i \in \mathcal{D}$, $Y_0^{(i)} = \frac{1}{2}e_i$, $s_0^{(i)} = \emptyset$.

- Iteration step, for all $n \geq 1$:

$$Y_n^{(i)} = \begin{cases} \mathrm{sym}_{Y_{n-1}^{(\delta_n)}}(X_{n-1}) & \text{if } i = \delta_n; \\ Y_{n-1}^{(i)} & \text{otherwise.} \end{cases} \qquad s_n^{(i)} = \begin{cases} \mathrm{sym}_{Y_{n-1}^{(\delta_n)}}(S_{n-1}) & \text{if } i = \delta_n; \\ s_{n-1}^{(i)} & \text{otherwise.} \end{cases}$$

$$X_n = Y_{n-1}^{(\delta_n)}, \qquad S_n = S_{n-1} \cup \mathrm{sym}_{Y_{n-1}^{(\delta_n)}}(S_{n-1})$$

Theorem 1 ([DV12])**.** *For all $n \geq 0$, we have:*

(i) S_n is symmetric about X_n and $s_n^{(i)}$ is symmetric about $Y_n^{(i)}$, which means $S_n = \mathrm{sym}_{X_n}(S_n)$ and $s_n^{(i)} = \mathrm{sym}_{Y_n^{(i)}}(s_n^{(i)})$.
(ii) The sets $s_n^{(1)}, s_n^{(2)}, \ldots, s_n^{(d)}$ are all connected, either empty or adjacent to 0 and pairwise non-adjacent,
(iii) $S_n = \{0\} \cup s_n^{(1)} \cup \cdots \cup s_n^{(d)}$.
(iv) The graph of S_n is a tree.

See Figure 2 and 3 for some examples of sets S_n. Note that in these figures, as it is usually the case when working with discrete planes, a points of $x \in \mathbb{Z}^d$ is displayed as a voxel centered in x.

We may now formulate the main result of this paper.

Theorem 2. *For each $n \geq 0$, $|\mathrm{Pal}(S_n)| = |\mathrm{vert}(S_n)|$.*

The proof of this theorem is provided in Section 4. More precisely, Theorem 2 is a direct consequence of Lemma 16 where the existence of an explicit bijection between $\mathrm{vert}(S_n)$ and $\mathrm{Pal}(S_n)$ is established.

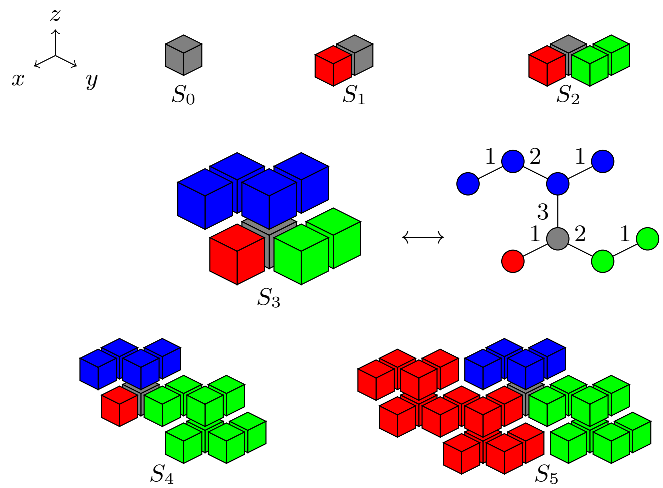

Fig. 2. Construction of the set S_5 for $\Delta = 12321\cdots$. The top row shows S_0, S_1, S_2. The middle row shows S_3 along with its labeled graph structure. The bottom row shows S_4 and S_5. The colors identify the sets $s_n^{(i)}$ while the origin is shown in gray. In each case, the origin is the only point connecting the subsets $s_n^{(i)}$. The palindromic language of S_3 is $\mathrm{Pal}(S_3) = \{\varepsilon, 1, 2, 3, 121, 131, 232, 12321\}$.

3.3 Construction by translations

The construction of the sequence $(S_n)_{n\geq 0}$ may equivalently be formulated in terms of translations. Indeed, call *symmetric* a set $U \subset \mathbb{R}^d$ such that there exists a point $x \in \mathbb{R}^d$ that satisfies $U = \mathrm{sym}_x(U)$, one checks that for any point $y \in \mathbb{R}^d$, we have $\mathrm{sym}_y(U) = U + t$, where $t = 2\,(y - x)$.

The sequence of translation vectors $(t_n)_{n\geq 1}$ is defined directly by the UFS algorithm. For each $n \geq 1$, let ω_n be the value of the coordinate of $\mathbf{v}^{(n)}$ that has been subtracted from the other ones. Using the notation $\sigma^\intercal$ for the transposed of matrix σ, we have

$$\begin{aligned}\omega_n &= \langle \mathbf{v}^{(n)}, e_{\delta_n} \rangle = \langle \sigma_{\delta_{n-1}} \mathbf{v}^{(n-1)}\ ,\ e_{\delta_n} \rangle = \langle \sigma_{\delta_{n-1}} \sigma_{\delta_{n-2}} \cdots \sigma_{\delta_1} \mathbf{v}\ ,\ e_{\delta_n} \rangle \\ &= \langle \mathbf{v}\ ,\ \sigma_{\delta_1}^\intercal \sigma_{\delta_2}^\intercal \cdots \sigma_{\delta_{n-1}}^\intercal e_{\delta_n} \rangle.\end{aligned}$$

The vector t_n is defined as $t_n = \sigma_{\delta_1}^\intercal \sigma_{\delta_2}^\intercal \cdots \sigma_{\delta_{n-1}}^\intercal e_{\delta_n}$ so that

$$\omega_n = \langle \mathbf{v}, t_n \rangle > 0. \tag{3}$$

Theorem 3 **([DV12], § 4).** *For each $n \geq 1$, $S_n = S_{n-1} \cup (S_{n-1} + t_n)$.*

Theorem 4 **([DV12], Lemmas 5 and 6).** *For each $k \in \mathcal{D}$, let n_k be the smallest index such that $\delta_{n_k} = k$, then*

$$t_1 + t_2 + \cdots + t_{n_k} = e_k.$$

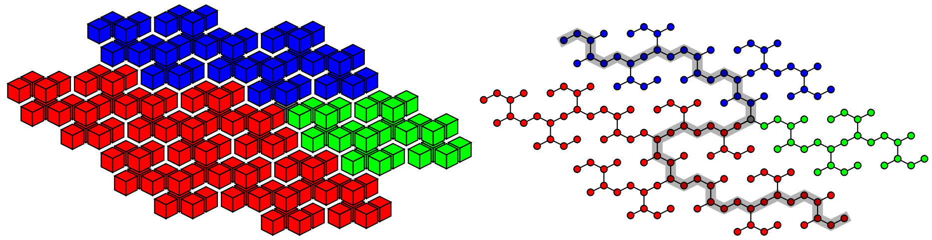

Fig. 3. On the left, the set S_8 for the directive sequence $\Delta = 12321231\cdots$. Subset $s_8^{(1)}$ is shown in red, $s_8^{(2)}$ in green and $s_8^{(3)}$ in blue. Note that, up to a translation, $s_8^{(2)}$ is equal to S_5 shown in Figure 2. On the right, the graph structure of S_8 where the longest palindromic path 1232121121232123231121211323212321211212321 is highlighted.

Moreover, let i and j be such that $\delta_i = \delta_j$ but for all $i < k < j$, $\delta_i \neq \delta_k$, then

$$t_{i+1} + t_{i+2} + \cdots + t_j = t_i.$$

Theorem 3 clearly implies that for each point $x \in S_\infty$ there exists $n \geq 0$ such that $x = \sum_{i=1}^{n} \varepsilon_i t_i$ where each $\varepsilon_i \in \{0,1\}$. We consider the function ψ that maps finite binary words to points of S_∞

$$\psi : \begin{cases} \{0,1\}^\star & \to S_\infty \\ \varepsilon_1 \cdots \varepsilon_n & \mapsto \sum_{i=1}^{n} \varepsilon_i t_i \end{cases}$$

The second part of Theorem 4 implies that ψ is not a bijection since more than one word ε may be mapped to the same point.

Lemma 5 ([DV12], Th. 4). *For all $n \geq 0$, $S_n = \{\psi(\varepsilon) \mid \varepsilon \in \{0,1\}^n\}$.*

3.4 More about S_n

We now introduce some technical properties of S_n. For each $n \geq 0$, let H_n be a designated point of S_n defined by $H_n = \psi(1^n)$.

Lemma 6. *For each $n \geq 0$, we have*

(i) *if $x \in \mathcal{S}_n$, then $x \neq 0 \Rightarrow \langle x, \mathbf{v}\rangle > 0$,*
(ii) *if $x \in \mathcal{S}_n$, then $x \neq H_n \Rightarrow \langle x, \mathbf{v}\rangle < \langle H_n, \mathbf{v}\rangle$,*
(iii) $H_n = \mathrm{sym}_{X_n}(0)$,
(iv) $H_{n+1} \in S_{n+1} \setminus S_n$.

Proof. Properties (*i*) and (*ii*) are consequences of Equation (3). Indeed, since $\omega_i > 0$ for all i, any word ε with at least one occurrence of the letter 1 is such that $\langle \psi(\varepsilon), \mathbf{v}\rangle > 0$. Similarly, any word ε' with at least one occurrence of the letter 0 is such that $\langle \psi(\varepsilon'), \mathbf{v}\rangle < \sum_{i=1}^{n} \omega_i = \langle H_n, \mathbf{v}\rangle$.

For property (*iv*), since $\langle H_{n+1}, \mathbf{v}\rangle = \langle H_n, \mathbf{v}\rangle + \omega_{n+1}$ and $\omega_{n+1} > 0$, we have $H_{n+1} \notin S_n$. Otherwise it would contradict property (*ii*).

Finally, for property (*iii*), the case $n = 0$ is trivial. For $n \geq 1$, by property (*iv*), there exists $x \in S_{n-1}$ such that $H_n = \mathrm{sym}_{X_n}(x)$. We have

$$\langle H_n, \mathbf{v} \rangle = \langle 2X_n - x, \mathbf{v} \rangle = 2 \langle X_n, \mathbf{v} \rangle - \langle x, \mathbf{v} \rangle ,$$

By property (*ii*), H_n is maximal in the sense that $\langle H_n, \mathbf{v} \rangle = \max\{\langle y, \mathbf{v} \rangle \mid y \in S_n\}$ which implies that x in minimal in the sense that $\langle x, \mathbf{v} \rangle = \min\{\langle y, \mathbf{v} \rangle \mid y \in \mathcal{S}_{n-1}\}$. So, by property (*i*), we have $x = 0$. □

4 Proof of the main theorem

The four following Lemmas provide a proof to Theorem 2. A path in S_n in coded by a word where each letter k codes a movement by e_k or $-e_k$. Thus, in general, the word coding a path alone does not contain enough information to retrieve the path itself. Nevertheless, when working in a set S_n, the structure of S_n is restrictive enough to partially retrieve this information.

Lemma 7. *Let $i, j \in \mathcal{D}$ and $x \in S_n$, then $i \neq j \Rightarrow x + e_i + e_j \notin S_n$.*

Proof. Let $\mathbf{v}_{i_1} \leq \mathbf{v}_{i_2}$ be the two smallest coordinates of $\mathbf{v}$. At the kth iteration of the UFS algorithm, the value ω_k is subtracted from each coordinate of $\mathbf{v}^k$ except one. In particular, it is either subtracted to $\mathbf{v}^k_{i_1}$, to $\mathbf{v}^k_{i_2}$ or both. Since $\mathbf{v} \in \mathcal{F}_d$, for all $n \geq 1$, all coordinates of $\mathbf{v}^n$ are strictly positive and therefore:

$$\mathbf{v}_{i_1} + \mathbf{v}_{i_2} > \sum_{i=1}^{n} \omega_i. \tag{4}$$

Suppose, by contradiction, that the above inequality is not respected then we have either $\mathbf{v}^{(n+1)}_{i_1} \leq 0$ or $\mathbf{v}^{(n+1)}_{i_2} \leq 0$.

Now, since $x \in S_n$, Lemma 5 implies that $0 \leq \langle x, \mathbf{v} \rangle \leq \sum_{i=0}^{n} \omega_i$ for all $x \in S_n$. On the other hand, the point $x + e_i + e_j$ is such that

$$\langle x + e_i + e_j, \mathbf{v} \rangle = \langle x, \mathbf{v} \rangle + \mathbf{v}_i + \mathbf{v}_j > \sum_{i=1}^{n} \omega_i,$$

which completes the proof. □

Lemma 8. *Let $w \in \mathcal{L}(S_n)$, there exists a constant γ_w such that for all pairs of points $x, y \in S_n$ with $\pi_{S_n}(x, y) = w$, we have $| \langle x, \mathbf{v} \rangle - \langle y, \mathbf{v} \rangle | = \gamma_w$.*

Proof. Lemma 7 implies that the sign of two consecutive steps coded by different letters must have different signs.

On the other hand, if $w_k = w_{k+1}$ then we must have $w_k = i_1$ and this time the sign may not change since, in a simple path, a movement e_{i_1} may not be followed by a movement $-e_{i_1}$.

Finally, the above argumentation states that the sign of each step of a path in S_n is completely determined by the sign of the first step. □

A consequence of the previous lemma is that in S_n, the word that codes the path from 0 to H_n does not appear anywhere else.

Lemma 9. *For all $x, y \in S_n$, we have*

$$\pi_{S_n}(x,y) = \pi_{S_n}(0,H_n) \Rightarrow \{x,y\} = \{0,H_n\}.$$

Proof. By construction, $\langle 0, \mathbf{v}\rangle$ is minimal and $\langle H_n, \mathbf{v}\rangle$ is maximal among all points of S_n. The result is direct from Lemma 8 and Lemma 6 (*i*) and (*ii*). $\square$

The point H_n is the key to understand the language $\mathcal{L}(S_n)$. This point plays a crucial role because it can be seen as the *gateway* between $S_n \setminus S_{n-1}$ and S_{n-1}. We show that H_n is the only point of $S_n \setminus S_{n-1}$ that is adjacent to a point of S_{n-1}.

Lemma 10. *For all $n \geq 1$, there exist a point $x \in S_{n-1}$ such that $H_n - x = e_{\delta_n}$.*

Proof. Let $(q_j)_j$ be the ordered sequence of the positions of all occurrences of the letter δ_n in Δ and let k be such that $q_k = n$. That is, for each i from 1 to n, there exists j with $1 \leq j \leq k$ such that $q_j = i$ if and only if $\delta_i = \delta_n$.

We build three words $w_e, w_x, w_h \in \{0,1\}^n$. First, consider the word 1^{q_1}, by Theorem 4 we have $\psi(1^{q_1}) = e_{\delta_n}$. If we replace the last occurrence of the letter 1 in 1^{q_1} with $0\,1^{q_2-q_1}$ then, again by Theorem 4, we have $\psi(1^{q_1-1}\,0\,1^{q_2-q_1}) = e_{\delta_n}$. By iterating this operation $k-1$ times, we obtain a word w_e of length n, as shown below, such that $\psi(w_e) = e_{\delta_n}$. Let w_x be the word obtained by applying the morphism that maps 0 to 1 and 1 to 0 to the word w_e so that $w_x = 0^{q_1-1}\,1\,0^{q_2-q_1-1}\,1\cdots$ and finally, let $w_h = 1^n$.

$$\begin{array}{rl} & 1\,2\,3 \quad \cdots \quad q_1 \quad \cdots \quad q_2 \quad \cdots \quad q_k \\ w_e = & 1\;1\;1\;11\cdots 11\;\;0\;\;11\cdots 11\;\;0\;\;11\cdots 11\;\;1 \\ w_x = & 0\;0\;0\;00\cdots 00\;\;1\;\;00\cdots 00\;\;1\;\;00\cdots 00\;\;0 \\ w_h = & 1\;1\;1\;11\cdots 11\;\;1\;\;11\cdots 11\;\;1\;\;11\cdots 11\;\;1 \end{array}$$

Let $x = \psi(w_x)$, we conclude by Lemma 5 that $x \in S_{n-1}$ and $x + e_{\delta_n} = H_n$. $\square$

Lemma 11. *The word $\pi_{S_n}(0, H_n)$ is a palindrome.*

Proof. Let $w = w_1 \cdots w_k = \pi_{S_n}(0,H_n)$, let $p = [p_0, p_1, \ldots, p_k] = P_{S_n}(0,H_n)$ and for each letter w_i, let $\varepsilon_i \in \{-1, 1\}$ be such that $p_i - p_{i-1} = \varepsilon_i e_{w_i}$.

We build a path $p' = [p'_0, p'_1, \ldots, p'_k]$ from H_n to 0 that is coded by the same word w then, since S_n is a tree we conclude that p' must be p read backwards and that w is a palindrome.

Let $p'_0 = H_n$, and let $p'_1 = p'_0 - \varepsilon_1 e_{w_1}$. We need to show that $p'_1 \in S_n$. By construction, S_n is invariant by sym_{X_n} so that $p'_1 \in S_n \iff \mathrm{sym}_{X_n}(p'_1) \in S_n$. We have:

$$\mathrm{sym}_{X_n}(p'_1) = \mathrm{sym}_{X_n}(H_n - \varepsilon_1 e_{w_1}) = 2X_n - H_n + \varepsilon_1 e_{w_1} = \mathrm{sym}_{X_n}(H_n) + \varepsilon_1 e_{w_1} = p_1.$$

Note that the last equality is obtained by Lemma 6 (*iii*). By doing the same for each p'_i with i from 2 to k we obtain the desired path p' from H_n to 0 which concludes this proof. $\square$

The following technical lemma shows that $P_{S_n}(0, H_n)$ is an obligatory passage for many paths in S_n.

Lemma 12. *For all $n \geq 1$, given $x \in S_n \setminus S_{n-1}$ and $y \in S_{n-1} \setminus \mathrm{sym}_{X_n}(S_{n-1})$, the path $P_{S_n}(x, y)$ passes by 0 and H_n.*

Proof. First we show that the sets $S_n \setminus S_{n-1}$ and $S_{n-1} \setminus \mathrm{sym}_{X_n}(S_{n-1})$ are connected. By the construction of S_n, we have $s_n^{(\delta_n)} = \mathrm{sym}_{X_n}(S_{n-1})$. Theorem 1 (*ii*) states that each $s_{n-1}^{(i)}$ is connected and adjacent to 0 so that $S_{n-1} \setminus s_{n-1}^{(i)}$ is connected. Note that $0 \notin s_{n-1}^{(i)}$ for all i since otherwise they would be pairwise adjacent. Then, also by the construction of S_n, we have

$$\begin{aligned} S_n \setminus S_{n-1}, &= \left(S_{n-1} \cup \mathrm{sym}_{X_n}(S_{n-1})\right) \setminus S_{n-1}, \\ &= \mathrm{sym}_{X_n}(S_{n-1}) \setminus S_{n-1}, \\ &= \mathrm{sym}_{X_n}\left(S_{n-1} \setminus \mathrm{sym}_{X_n}(S_{n-1})\right), \\ &= \mathrm{sym}_{X_n}\left(S_{n-1} \setminus s_{n-1}^{(i)}\right). \end{aligned}$$

Note that the third equality uses the fact that sym_{X_n} is an involution. It is then obvious that both set $S_n \setminus S_{n-1}$ and $S_{n-1} \setminus \mathrm{sym}_{X_n}(S_{n-1})$ are connected.

By contradiction, suppose that there exists a path p_x from x to y that does not passes by H_n. Lemma 10 shows that H_n is adjacent to a point of S_{n-1}. Since S_{n-1} is a tree, there exist a path p_h from H_n to y such that every points in this path, except H_n, is in S_{n-1}. Since $S_n \setminus S_{n-1}$ is connected there exists a path p from x to H_n that contains no points of S_{n-1}. The paths p_x, p_h and p form a non-trivial loop which is impossible since S_n is a tree. We have shown that H_n must be in $P_{S_n}(x, y)$. The fact that $0 \in P_{S_n}(x, y)$ is shown similarly. □

4.1 A bijection using palindromic closure

We now define a bijection from the points of S_n to the set of its palindromes. First we need the notion of *palindromic closure* of words which was introduced by de Luca [dL97] for the study of Sturmian words.

Definition 13. *Given a word w, its* palindromic closure *w^+ is the shortest palindrome such that w is a prefix of w^+.*

Let $\mathrm{lps}(w)$ denote the longest palindromic suffix of the word w. It is well know that $w^+ = u \cdot \mathrm{lps}(w) \cdot \widetilde{u}$ where u is such that $w = u \cdot \mathrm{lps}(w)$ and $\widetilde{u}$ is the word u read backwards.

Definition 14. *Let* $\Phi : \begin{array}{l} S_n \to \mathrm{Pal} \\ x \;\mapsto \pi_{S_n}(x, 0)^+ \end{array}$

Lemma 15. *For all $x \in S_n$, there exists $y \in S_n$ such that $\pi_{S_n}(x, y) = \Phi(x)$.*

Proof. Without loss of generality, suppose that n is minimal in the sense that $x \notin S_{n-1}$. Let $u = \pi_{S_n}(x, H_n)$ and $v = \pi(H_n, 0)$. By Lemma 12 we have $\pi_{S_n}(x, 0) = uv$. We begin by showing that $v = \text{lps}(uv)$. Lemma 11 ensure that v is a palindrome while Lemma 9 and properties (i) and (ii) of Lemma 6 ensure that word v may not be read at any other place in S_n. Since a palindromic suffix that is not the longest one must appear at least twice, we conclude that $v = \text{lps}(uv)$.

Let $p = [p_0, p_1, \ldots, p_k] = P_{S_n}(x, H_n)$. Lemma 12 implies that $p_i \in S_n \setminus S_{n-1}$ for i from 0 to k. As a consequence, the path $p' = [\text{sym}_{X_n}(p_k), \ldots, \text{sym}_{X_n}(p_0)]$ is a path in S_{n-1} which is coded by the word $\widetilde{u}$. Finally, since $p_k = H_n$, we have that p' starts at 0 which concludes this proof. □

We complete the proof of Theorem 2 by showing that Φ is a bijection.

Lemma 16. *For each $p \in \text{Pal}(S_n)$ there is a unique $x \in S_n$ such that $\Phi(x) = p$.*

Proof. Let $m \leq n$ be the smallest integer such that $p \in \text{Pal}(S_m)$. First we consider the case where p is the empty word, then $x = 0$ is the only point such that $\Phi(x) = p$. Otherwise, let p be such that $|p| \geq 1$, then we have that $m \geq 1$. The minimality of m forces that for all $x, y \in S_{n_p}$ such $\pi_{S_m}(x, y) = p$, then x and y cannot both be in S_{m-1} since otherwise we would have $p \in \text{Pal}(S_{m-1})$. Similarly, for the same reason x and y cannot both be in $\text{sym}_{X_{m-1}}(S_{m-1})$.

Without loss of generality, suppose that $x \in \text{sym}_{X_{m-1}}(S_{m-1}) \setminus S_{m-1}$ and $y \in S_{m-1} \setminus \text{sym}_{X_m-1}(S_{m-1})$.

By Lemma 12 we have that $p = uvw$ where $v = \pi_{S_m}(0, H_m)$. We have already seen that $\pi_{S_m}(0, H_m)$ appears only once in S_m, this implies that v must be in the center of p since otherwise it would be repeated. The palindrome p can be written $p = uv\widetilde{u}$.

Now, by contradiction, suppose there are two points x and x' in $S_m \setminus S_{m-1}$ such that $w_{S_m}(x, H_n) = w_{S_m}(x', H_n)$. Let $p = [p_0, p_1, \ldots, p_k] = P_{S_m}(H_n, x)$, there exist ℓ with $1 \leq \ell \leq k$ such that $P_{S_m}(H_n, x') = [p_0, p_1, \ldots, p_{\ell-1}, p'_\ell, \ldots, p_k]$ with $p_\ell \neq p'_\ell$. Since both paths are coded by the same word, we have

$$p_\ell - p_{\ell-1} = -(p'_\ell - p_{\ell-1}) = \pm e_{u_{\ell+1}}.$$

Let z be the point before $p_{\ell-1}$ in the path $P_{S_m}(0, x)$ (which is $p_{\ell-2}$ if $\ell \geq 2$). Suppose that $p_{\ell-1} - z = e_i$ for some $i \in \mathcal{D}$ (the other possibility being $p_{\ell-1} - z = -e_i$ which is similar). If $i = u_\ell$ then one of the paths $P_{S_m}(0, x)$ or $P_{S_m}(0, x')$ makes a movement back and forth and this is impossible in a simple path. We are left to consider the case $i \neq u_\ell$, in this case we have that either $p_\ell = z + e_i + e_{u_\ell}$ or $p'_\ell = z + e_i + e_{u_\ell}$ but both cases are impossible according to Lemma 7. □

5 Conclusion

We have considered a construction of discrete hyperplanes guided by the fully subtractive multidimensional continued fraction algorithm. This specific construction builds finite sets with a tree structure which, for well chosen normal vectors, covers the whole discrete plane. We have provided a complete proof that the number of palindromes in the language of these trees is equal to its number of vertices. This results generalizes the one by Droubay, Justin and Pirillo who showed that the number of palindromes in a finite Sturmian word is equal to its length plus one. This bound is known to be maximal for words [DJP01,BHNR04].

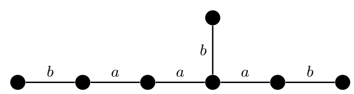

Fig. 4. A tree $\mathcal{T}$ with $\text{vert}(\mathcal{T}) = 7$ vertices and $|\text{Pal}(\mathcal{T})| = 8$.

It is worth mentioning that unlike linear words, trees may contain more palindromes than the number of letters plus one which correspond to the number of vertices in the tree. Figure 4 illustrates a tree $\mathcal{T}$ for which $|\text{Pal}(\mathcal{T})| > |\text{vert}(\mathcal{T})|$.

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