Graph encoding of 2D-gon tilings

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Abstract

2D-gon tilings with parallelograms are a model used in physics to study quasicrystals, and they are also important in combinatorics for the study of aperiodic structures. In this paper, we study the graph induced by the adjacency relation between tiles. This relation can been used to encode simply and efficiently 2D-gon tilings for algorithmic manipulation. We show for example how it can be used to sample random 2D-gon tilings.

Key words: tilings, algorithmics, graph theory

1 Introduction

A tiling can be defined as a partition of a given region of an affine space. More classically, one considers a finite set of shapes, called *prototiles*, and a region of an affine space. The tiling problem is then to decide whether this region can be tiled, i.e. covered by translated copies of prototiles, without gaps or overlappings between them. If this is possible, the region is *tilable*,

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and a solution is called a *tiling* of the region. The translated copies of the prototiles are the *tiles* of the tiling. If the region to tile is the whole plane, this problem has been shown to be undecidable by Berger [Ber66], which was the first important incursion of tilings in computer science.

In this paper, we are concerned with tilings of $2D$-gons with parallelograms. A $2D$-gon is an hexagon when D=3, an octagon when D=4, a decagon when D=5, etc. Such a region can always be tiled with parallelograms. $2D$-gon tilings by parallelograms appear in physics as a model for quasicrystals [Des97] and aperiodic structures [Sen95]. They are also used to encode several combinatorial problems [Eln97,Lat00], and have been studied from many points of view [Lat00,RGZ94,Bai99]. In particular, they are strongly related to the oriented matroid theory, since the Bohne-Dress theorem proves the equivalence of $2D$-gon tilings with a class of oriented matroids [RGZ94,BVS$^+$99].

These tilings cannot be easily manipulated by a program when one uses the geometric definitions. Some efficient solutions arise from the oriented matroids side [BFF01,CFG00]. We propose here another solution from graph theory, which has the advantage of giving very simple algorithms and to introduce some interesting questions about graphs related to 2D-gon tilings. The aim of this paper is to study the graph induced by the adjacency relation between tiles, and to find the minimal amount of information that needs to be added to this graph in order to provide an effective notion of dual graph of a tiling. Our results prove that this information can be reduced to the knowledge of two tiles which are in successive position on the hull of the zonotope. In particular, the knowledge of vector and de Bruijn line multiplicities is not required and can be computed.

We will first present the tilings more formally, and define the adjacency graph we will use. Then we will obtain a one-to-one correspondence between a class of graphs and $2D$-gon tilings by introducing the notion of *graph with origins*, and we give an algorithm which builds the $2D$-gon tiling corresponding to a given graph with origins. We will finally see how the flip operation can be defined on the graph, which makes it possible to sample random tilings of $2D$-gons. Let us emphasize on the fact that we have two aims in this study: give some properties of the adjacency graph of $2D$-gon tilings, which is a fundamental object on which very few information is known, and provide a very simple and efficient way to algorithmically manipulate 2D-gon tilings.

2 Preliminaries

Given two vectors v and v', we will say that $v < v'$ in the *natural order* if the angle between $(1,0)$ and v is smaller than the angle between $(1,0)$ and v'.

Let $V = \{v_1, v_2, ..., v_D\}$ be a family of D pairwise non collinear vectors with nonnegative second component of the plane. We suppose that for all integer i, $v_i < v_{i+1}$ in the natural order. Let $M = \{m_1, m_2, ..., m_D\}$ be a family of D positive integers. The integer m_i is called the *multiplicity* of v_i. The $2D$-gon P associated with V and M is the region of the affine plane defined by:

$$\left\{ \sum_{i=1}^{D} \lambda_i v_i, 0 \leq \lambda_i \leq m_i, m_i \in M, v_i \in V \right\}$$

There exists many equivalent definitions for these objects. For example, a $2D$-gon can be viewed as the projection of a hypercube of dimension D onto the plane. See [Zie95] for more details. For D=2, the $2D$-gons are parallelograms; for D=3, hexagons; for D=4, octagons; for D=5, decagons; etc. See Figure 1 for an illustration.

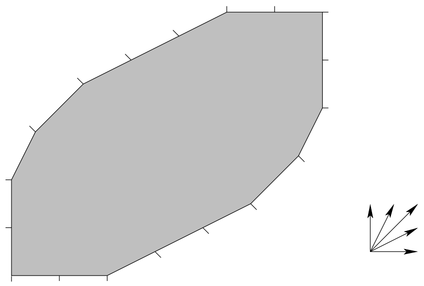


Figure 1. A $2D$-gon and its vectors, associated with $M = \{2, 3, 1, 1, 2\}$. This is a decagon with sides of length 2,3,1,1,2,2,3,1,1,2.

Given a $2D$-gon P, a prototile of $P = (V, M)$ is a $2D$-gon built using only 2 vectors in V, each of them with multiplicity 1. Therefore, each prototile of P is a parallelogram defined by two vectors in V, and we will make no distinction between the prototile viewed as an area and the pair of the indices of the vectors in V which define it. A side of a tile will be called *edge*, and the vector defining an edge is the *support* of the edge.

Finally, a tiling T of a $2D$-gon $P = (V, M)$ is a set of tiles (i.e. translated copies of the prototiles) which cover exactly P and such that there is no overlapping between tiles. Therefore, T is a set of couples, their first component being the pair of vectors which defines the prototile, the second one being a translation, or position. The positions used in $2D$-gon tilings can always be written as a linear combination of vectors in V : $t = \sum_i t_i v_i$, t_i being an integer between 0 and m_i. More precisely, a point p of the tiling can be positioned according to each vector by being assigned the (algebraic) number of occurrences of this vector one has to use in a direct path from the point $(0, 0)$ to p along the edges of the tiling. This leads to a D-dimensional vector, whose components are the positions according to each vector of V. A tile is assigned the position of its point closest to $(0, 0)$.

Therefore, the position will be described by the D-dimensional vector $(t_1, ..., t_D)$. Two tilings T and T' of two $2D$-gons P and P' are said to be *equivalent* if $T = T'$, where T and T' are viewed as sets of couples.

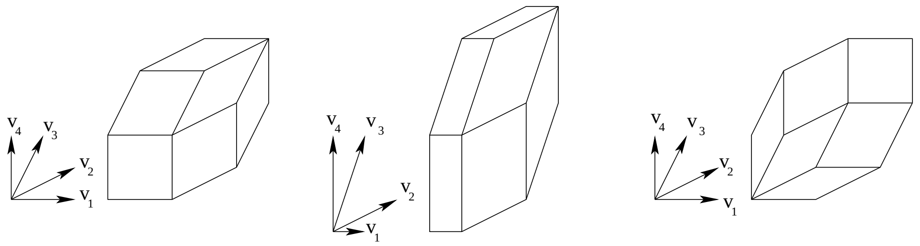


Figure 2. Three tilings of $2D$-gons, namely T_1, T_2 and T_3 from left to right; T_1 and T_2 are *equivalent*, whereas T_3 is equivalent to none of the others.

Consider for example the three tilings in Figure 2. From left to right, they are described by :

$$T_1 = \left\{ \begin{array}{l} (\{1,2\},(0,0,1,1)),(\{1,3\},(0,0,0,1)),(\{1,4\},(0,0,0,0)),(\{2,3\},(1,0,0,1)), \\ (\{2,4\},(1,0,0,0)),(\{3,4\},(1,1,0,0)) \end{array} \right\}$$

$$T_2 = \left\{ \begin{array}{l} (\{1,2\},(0,0,1,1)),(\{1,3\},(0,0,0,1)),(\{1,4\},(0,0,0,0)),(\{2,3\},(1,0,0,1)), \\ (\{2,4\},(1,0,0,0)),(\{3,4\},(1,1,0,0)) \end{array} \right\}$$

$$T_3 = \left\{ \begin{array}{l} (\{1,2\},(0,0,0,0)),(\{1,3\},(0,1,0,0)),(\{1,4\},(0,1,1,0)),(\{2,3\},(0,0,0,0)), \\ (\{2,4\},(0,0,1,0)),(\{3,4\},(0,0,0,0)) \end{array} \right\}$$

Therefore, tilings T_1 and T_2 are equivalent, while T_1 and T_3 are not.

Let P be a $2D$-gon, and T be a tiling of P. The i-th *de Bruijn family* of T is the set of all the tiles in T which are built with the vector v_i. Moreover, each family can be decomposed into *de Bruijn lines*: the j-th de Bruijn line of the i-th family is the set of tiles built with v_i which have $j-1$ as the i-th component of their position. Continuing with our example of Figure 2, we obtain that the first line of the second de Bruijn family is equal to $\{(\{2,4\},(1,0,0,0)),(\{2,3\},(1,0,0,1)),(\{2,1\},(0,0,1,1))\}$ for T_1 and T_2. For practical convenience, we will also say that the j-th line of the i-th family is the α-th line of the tiling where $\alpha = \sum_{k=1}^{i-1} m_k + j$, m_k being the multiplicity of the k-th family. We also define $f(\alpha)$ as the index of the vector associated with the α-th line, i.e. $f(\alpha)$ is the number of the line's family. See Figure 3 for example. Notice that two lines in the same family never have a tile in common, whereas two lines in two different families always have exactly one tile in common. Moreover, each line l, j-th line of the i-th family, divides the set of tiles which are not in l into two disjoint parts, one having tiles whose i-th component is more than j, and the other having tiles whose i-th component is less. We will use these classical properties (see [dB81]) in the following.

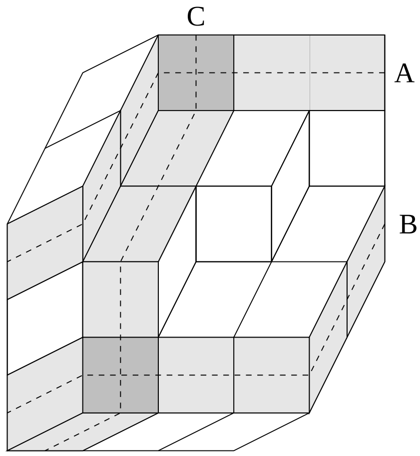


Figure 3. A $2D$-gon tiling and three de Bruijn lines (a line is a set of tiles crossed by a dotted line). A and B are in the same de Bruijn family. A is the third line of the fourth family, and B the second line of the same family. C is the first line of the first family. According to our notations, A is the 9-th line of the tiling, B is the 7-th, and C is the first. Notice that C crosses exactly once A and B, whereas A and B do not cross each other.

Before entering in the core of the paper, we need a few more notations, which we introduce now.

Definition 1 *Let $P = (V, M)$ be a $2D$-gon. For all k, $1 \leq k \leq D$ we define the k-th* side *of P as the set of points:*

$$\left\{\sum_{i=1}^{D} \lambda_i v_i, \forall i < k : \lambda_i = m_i, \forall i > k : \lambda_i = 0, \textit{and } 0 \leq \lambda_k \leq m_i\right\}$$

Likewise, we define the $(k+D)$-th side of P as:

$$\left\{\sum_{i=1}^{D} \lambda_i v_i, \forall i > k : \lambda_i = m_i, \forall i < k : \lambda_i = 0, \textit{and } 0 \leq \lambda_k \leq m_i\right\}$$

Moreover, the hull *of P, denoted by $H(P)$, is the union of all the sides of P. We will also say that a tile t is on the i-th side of P if one of the edges of t is included in the i-th side of P.*

Finally, each side i of P can be divided into copies of the vector v_i defining i, called steps, *and orientating i in the sense of v_i, one can order the edges of type v_i. Then one can define the j-th tile on the side i as being the tile having an edge on the j-th step of the side i.*

We can now introduce the notion of adjacency graph associated with a tiling, which will be the main object of our study.

Definition 2 (adjacency graph of a tiling) *Let T be a tiling of a given $2D$-gon (V, M), and let $n = |T|$ be the number of tiles of this tiling. The adjacency graph of T is the undirected graph $A(T) = (V_T, E_T)$ where $V_T = \{t \in T\}$ and $\{t, t'\} \in E_T$ if and only if t and t' have one edge in common in T. See Figure 4 for an example.*

Notice that the adjacency graph has all vertices with degree 4, except the vertices corresponding to the tiles on the hull of the tiling, which have degree 2 or 3. In the following, we consider tiles and their corresponding number without distinction.

Definition 3 (geodesic, distance, length of a path) *Let T be a tiling, A its adjacency graph, and t, t' two tiles of T. A* path *from t to t' is a sequence of tiles $t_0, .., t_r$ such that t_i and t_{i+1} are adjacent, $t_0 = t$ and $t_r = t'$. Such a path, containing $r+1$ vertices, has length r. Moreover, the sequence of vertices of A corresponding to the path is the path from t to t' in A.*
A geodesic *from t to t' is a shortest path from t to t'. The* distance *between t and t', denoted by $dist(t, t')$ is the length of the shortest path between t and t'.*

The adjacency graphs of $2D$-gon tilings will be our main object of interest in the rest of this paper. We will see that they encode much information on the tiling. However, the fact that two tilings have the same adjacency graph does not imply that they are equivalent: for example, one can verify that the tilings T_1 and T_3 in Figure 2 have the same adjacency graph. In order to obtain a one-to-one correspondence between a set of graphs and the set of tilings of a $2D$-gon, we introduce now the *de Bruijn graph*.

Definition 4 (de Bruijn graph of a tiling) *Let $A = (V, E)$ be the adjacency graph of a tiling T of a $2D$-gon P. The* de Bruijn graph *$A' = (V, E, \lambda, \nu)$ is a graph with labeled vertices and with a distinguished vertex ν. The label $\lambda(t)$ of $t \in V$ is the pair of integers $\{\alpha, \beta\}$ such that the two de Bruijn lines which contain the tile t are the $\alpha - th$ and $\beta - th$. The vertex ν, called the* origin *of the graph, is associated with the tile with translation vector $(0, ..., 0)$ which is on the first side of P. See Figure 4 for an example.*

Theorem 1 *Given the de Bruijn graph of a tiling T, Algorithm 1 constructs a tiling equivalent to T in time $O(n)$, where n is the number of tiles of T, i.e. the number of vertices of the graph.*

Proof: The idea of the algorithm is to start with the origin of the graph, and then make a breadth-first search which makes it possible to compute the tile associated with each vertex, i.e. the couple of vectors which describes the prototile, and the translation vector.

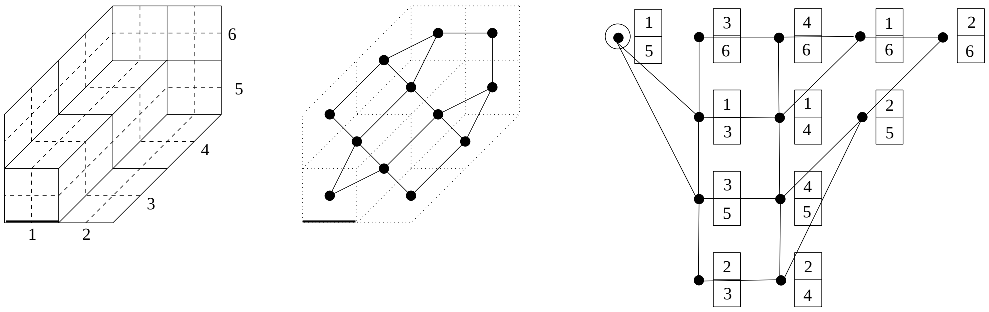


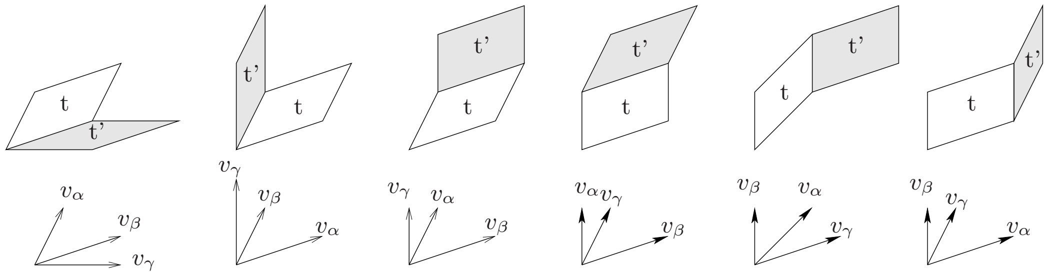
Figure 4. A tiling, its adjacency graph and its de Bruijn graph. Bold is the first side, determining the origin.

Let us consider a vertex v labeled with $\{\alpha, \beta\}$, which means that it corresponds to a tile $t = (\tau, \text{trans})$ in the α-th de Bruijn line and the β-th one. Recall that $f(x)$ is the number of the de Bruijn family of the line x. Then the prototile τ is defined by $f(\alpha)$ and $f(\beta)$. The function f is easy to compute, since two de Bruijn lines α and β are in the same family if and only if they do not cross each other, i.e. if there is no vertex labeled $\{\alpha, \beta\}$ in the de Bruijn graph. Therefore, we can easily find all the lines which belong to the same family, and so we can find the de Bruijn families. Finally, we obtain the prototile τ.

The next point is to compute the translation vectors. Each of them is deduced from the translation vector of a previously marked vertex. Since we start with a vertex with translation vector $(0, ..., 0)$ (the origin), and since we visit the vertices in a breadth-first order, a neighbor v' of v visited after v has a translation vector componentwise greater than or equal to the one of v. Suppose that we have already computed the tile $t = (\tau, \text{trans})$ which corresponds to v, and consider $t' = (\tau', \text{trans'})$ which corresponds to v'. As discussed above, we already have $t = \{\alpha, \beta\}$ and $t' = \{\beta, \gamma\}$ and trans' $\geq$ trans. The six cases illustrated in Figure 5 can occur. They lead to two possibilities :

- if $f(\alpha) > f(\beta) > f(\gamma)$ (see Figure 5.a) or $f(\gamma) > f(\beta) > f(\alpha)$ (see Figure 5.b), then trans' = trans
- in the other cases (see Figure 5.c,d,e,f), the α-th component of the translation vector has to be increased by one.

These remarks lead directly to Algorithm 1, and since we visit each vertex twice (once to compute $f(x)$ for all x, and once in the main loop), its complexity is $O(n)$, where n is the number of vertices. $\square$



(a) (b) (c) (d) (e) (f)

Figure 5. The six possible positions of the two tiles t and t' during the computation of the translation vectors.

Algorithm 1: Construction of a tiling from its de Bruijn graph.

Input: $G = (V, E, \lambda, \nu)$, the de Bruijn graph of a tiling T.

Output: A tiling equivalent to T, given by a list of $(prototile, translation)$.

begin

  Let $\{\alpha, \beta\} = \lambda(\nu)$;
  Set all the vertices as unmarked;
  $resu \leftarrow \{(\{f(\alpha), f(\beta)\}, (0, ..., 0))\}$;
  current $\leftarrow \{(\nu, (0, 0, ..., 0))\}$;
  Mark ν ;
  while *current* $\neq \emptyset$ **do**
    foreach $\tau = (v, trans)$ *in current* **do**
      foreach *unmarked vertex* v' *such that* $(v, v') \in E$ **do**
        Let $\{\alpha, \beta\}$ be the label of v, and $\{\beta, \gamma\}$ be the label of v';
        Let trans' be a copy of trans;
        if *not* $(f(\alpha) > f(\beta) > f(\gamma)$ *or* $f(\gamma) > f(\beta) > f(\alpha))$ **then**
          Increase the $f(\alpha)$-th component of trans' by one;
        resu $\leftarrow$ resu $\cup \{(\{f(\beta), f(\gamma)\}, \text{trans'})\}$;
        current $\leftarrow$ current $\cup \{(v', \text{trans'})\}$;
        Mark v';
      current $\leftarrow$ current $\setminus \{v\}$;
  Return (resu);

end

This result shows that all the information contained in a $2D$-gon tiling is encoded in its de Bruijn graph. However, we will show that the de Bruijn graph contains much more information than really needed to construct the tiling. Actually, we will show that the adjacency graph contains almost all the information we need. Indeed, it suffices to add two marks to the adjacency graph of T to be able to reconstruct the tiling T. This leads to the definition of the graph with origins of a tiling T.

Definition 5 (graph with origins of a tiling) *Let T be a tiling of a $2D$-gon, $A = (V, E)$ its adjacency graph. The* graph with origins *associated with T is $G = (V, E, v_1, v_2)$, where v_1 and v_2 are two vertices in V called the* origins *of G and defined as follows. v_1 is the tile of T on the first side of P with translation vector $(0, ..., 0)$. v_2 is the tile having an edge on the hull of P, adjacent to v_1 by a vertex on the hull of P and whose translation vector is either $(1, 1, 0, ..., 0)$ if v_1 belongs also to the second side of the hull of P, or $(1, 0, ..., 0)$ otherwise. See Figure 6 for an example.*

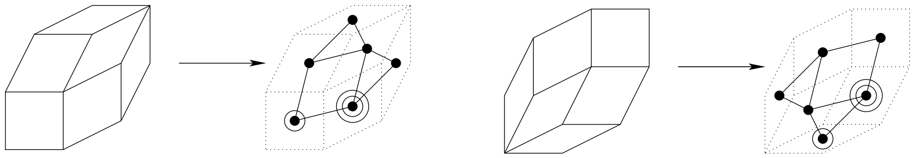


Figure 6. Two tilings and their graphs with origins. Notice that, if one removes the origins, then the two graphs are isomorphic.

Notice that the addition of the two origins makes it possible to distinguish two different tilings which have isomorphic adjacency graphs, as shown for example in Figure 6. We will show in the following that this is always true: the correspondence between the graphs with origins we defined and the $2D$-gon tilings is one-to-one.

3 Duality

In this section, we give an algorithm which computes the de Bruijn graph of a tiling from its graph with origins. This correspondence is one-to-one, therefore, together with Algorithm 1 and Theorem 1, it shows that the graphs we introduced can be considered as *dual* of the considered tilings, despite the fact that they are very close to adjacency graphs (they only have two additional marks). Our algorithm has complexity $O(n \cdot m)$, where n is the number of vertices of the graph, or equivalently the number of tiles of the tiling, and m is the sum of the multiplicities used to define the $2D$-gon.

In order to build the algorithm and prove its correctness, we will first prove some properties linking tilings of $2D$-gons and their adjacency graphs. In particular, some special sub-structures, namely *borders* and *fans*, will play a very important role. We introduce them now, and prove some of their basic properties.

Definition 6 (border of a tiling) *Let T be a tiling of a $2D$-gon P. Let $C \subseteq T$ be the set of tiles of T which have at least one point in $H(P)$. We define the* border *of T, denoted by $B(T) = (C, E)$, as follows: $(t, t') \in E$ if and only if t and t' have one edge in common and if this common edge has at least one* *point in $H(P)$. Notice that this is a subgraph of the adjacency graph of T, but it is not the subgraph* induced *by C (some edges are missing). See Figure 7.*

A remarkable property is that the border contains all vertices of degree less than 4 of the graph (i.e. degree 2 or 3), plus possibly some of the vertices of degree 4.

In the following, we make no difference between the border of a tiling T and the associated path in the adjacency graph of T, called the border of the graph.

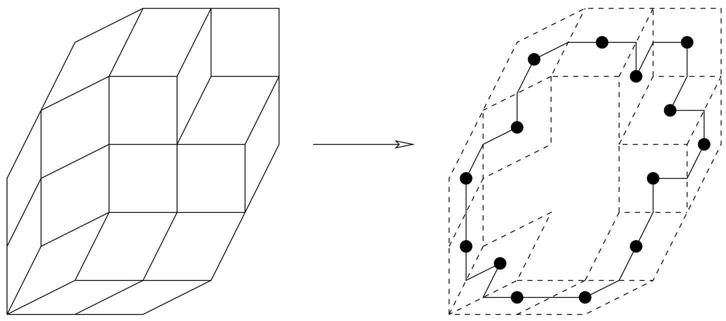


Figure 7. Left: a tiling. Right: its border and the border of its adjacency graph.

3.0.1 Properties of the border

Every vertex of degree 2 or 3 is in the border. Moreover, there are vertices whose degree is 4. Particular cases can occur. The border of a tiling can thus have 3 possible shapes :

- if there is a vertex whose degree for the border is 1, then $M = (1, n)$. The tiling is just a juxtaposition of tiles of the same type, and the border is a line. See Figure 8(left).
- if there is a unique vertex of degree 4, then $M = (1, 1, n)$, the tiling is an hexagon whose sides lengths are 1,1 and n. Then the border is the disjoint union of two cycles connected by the only vertex whose degree is 4. See Figure 8(right).
- in any other case, the border of the tiling is a cycle.

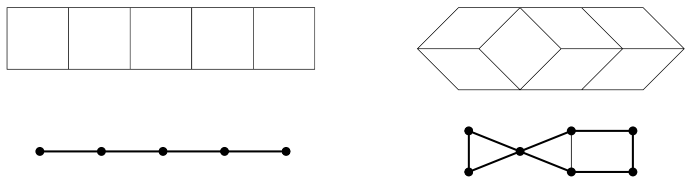


Figure 8. Particular cases and their adjacency graphs. Bold is the border.

The two first cases are easy to recognize: we are in the first one if and only if the adjacency graph is reduced to a path, we are in the second one if and only if the adjacency graph contains a cut node (i.e. a vertex whose deletion disconnects the graph). In those cases, the tiling is then easy to construct.

In the following, we assume that we are not in one of those (pathological) cases. There is no loss of generality, from the remarks just above.

Definition 7 (fan) *Given a tiling T of a $2D$-gon P, a* fan *F of T is a p-tuple $(f_1, f_2, .., f_p)$ of tiles in T such that:*

- $\forall i \in \{1, ..., p-1\}$ *f_i and f_{i+1} have an edge in common,*
- $\cap_{i=1}^{p} f_i$ *is a point of $H(P)$, called the* head *of the fan*
- *f_1 and f_p have one edge on the hull of the $2D$-gon.*

The tiles f_1 and f_p are called the endpoints *of F, and the integer $p-1$ is called the* length *of F. Moreover, we say that a fan F belongs to one side of P if one of its endpoints has an edge included in this side of P.*

Notice that two adjacent tiles on the hull having one edge in common constitute a particular fan of size 1.

Informally, a fan can be seen as the neighborhood of its head.

Remark: the border of a tiling is a concatenation of all the fans of the tiling. Moreover, the two origins given in the graph are the endpoints of a particular fan.

For convenience, we introduce some notations about fans.

Notation : Let $F = (f_1, ..., f_p)$ be a fan of a $2D$-gon tiling T. For all i such that $1 \le i \le p-1$, since f_i and f_{i+1} have exactly one edge in common, there is a de Bruijn line, denoted by L_i, which contains both f_i and f_{i+1}. In addition, each L_i can be split into two distinct parts, called *half-lines*, one containing f_i and the other containing f_{i+1}. We denote the part containing f_i by L_i^+, and the other one by L_i^-. See Figure 9.

Notice that L (respectively L_p), the Bruijn line different from L_1 (respectively L_{p-1}) passing in f_1 (respectively f_p), crosses L_1^+, L_2^+, $\ldots$, L_{p-1}^+, in the order of increasing (respectively decreasing) indices starting from the fan.

Finally, we remark that, starting from the fan, L_i^- successively crosses L_{i+1}^+, L_{i+2}^+, $\ldots$, L_{p-1}^+ and L_p, L_i^+ successively crosses L_{i-1}^-, L_{i-2}^-, $\ldots$, L_1^- and L. Thus, two half-lines having same sign do not cross each other.

Moreover, by construction, one may notice that, as a de Bruijn line, a half-line separates the set of tiles non contained in itself into two parts (formally, we

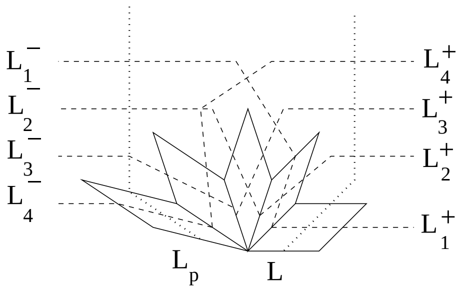


Figure 9. A fan, and its corresponding half-lines.

first define, for each tile t'' of L_i, the representing line segment of t'' as the closed line segment linking centers of edges of t'' whose type is $\{v_i\}$ (where v_i is the common vector defining the family of L_i). For L_i^+, consider the set S_i^+ formed by the union of representing segments of tiles of L_i^+ and the line segment common to f_i and f_{i+1}. By definition, two tiles t and t' (not in L_i^+) are in the same part if there exists a continuous function f from $[0,1]$ to the 2D-gon P such that, $f(0)$ is the center of the tile t, $f(1)$ is the center of the tile t', and for each real number $x \in [0,1]$, $f(x)$ is not element of the set S_i^+).

3.1 Shortest paths

We first define some material broadly used in the following proofs.

Definition 8 (line index, line value) *Let T be a tiling, L be a de Bruijn line of T or a half-line as above, and t_1, t_2 be two tiles of T. As seen before, L induces a partition of T into three parts T^+, T^- and L. We define the* line index *$ind_L(t_1, t_2)$ as follows:*

- *if $t_1 \in L$ and $t_2 \in T^+ \cup T^-$, then $ind_L(t_1, t_2) = 1/2$*
- *if $t_1 \in T^+$ and $t_2 \in T^-$, or $t_1 \in T^-$ and $t_2 \in T^+$, then $ind_L(t_1, t_2) = 1$*
- *in the other cases, $ind_L(t_1, t_2) = 0$*

Let $\boldsymbol{B} = \{L_1, ..., L_q\}$ be a set of de Bruijn lines and half-lines of T. The line value *$ind_{\boldsymbol{B}}(t_1, t_2)$ is the sum $\sum_{k=1}^{q} ind_{L_k}(, t_1, t_2)$.*

Intuitively, the line value from t_1 to t_2 counts the number of lines of $\mathbf{B}$ crossed by a path from t_1 to t_2.

Let $(t_0, t_1, \dots, t_r)$ be a path of tiles (i. e. two consecutive tiles are adjacent) and consider the sequence $(a_0, a_1, \dots, a_r)$ such that for each integer i, $0 \leq i \leq r$, $a_i = ind_{\mathbf{B}}(t_1, t_i)$ (See Figure 10). We assume that $\mathbf{B}$ is *correctly constructed* , i. e. for each half-line L^+, at most one element of $\{L^-, L^+, L\}$ is in $\mathbf{B}$, and if L^+ (or L^-) is in $\mathbf{B}$, then the endpoint of this half-line in the fan is not element of another line (or half-line) of **B**.

Remark that for each integer i such that $1 \leq i < r$, $ind_L(t_1, t_i) = ind_L(t_1, t_{i+1})$, except if L is a line containing one of the tiles t_i or t_{i+1} and not the other one. Since **B** is correctly constructed, in any case, there are at most two of these exceptions, for each of which the index can change of 1/2 unit. Thus, $|a_i - a_{i+1}| \leq 1$. Moreover we have $a_0 = 0$. Thus, we have:

- $r \geq a_r$, the line value is a lower bound for the distance between two tiles,
- the equivalence: $r = a_r$ if and only if for each integer i such that $0 \leq i \leq r$, $a_{i+1} - a_i = 1$.

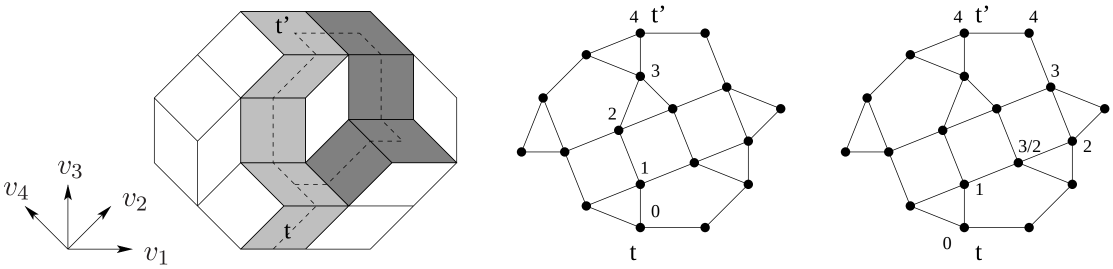


Figure 10. Two paths with same endpoints, and the values associated with each vertex, B being the set of all lines not in the family defined by v_1.

Lemma 1 *Let t and t' be two tiles of the border linked by a de Bruijn line L, and let $(t_0, t_1, \ldots, t_r)$ be the path of tiles starting in $t = t_0$ induced by L (i. e. $t_0 = t$, $t_r = t'$ and all tiles t_i are in L). The path $(t_0, t_1, \ldots, t_r)$ is the unique shortest path between t and t'.*

Proof: We take **B** as the set of the r de Bruijn lines which cut the line L. With the above notations, we have $a_r = r$ which proves that the path is a shortest path.

Now, assuming the shortest path from t to t' is not unique, let p be another path of length r from t to t'. The paths p and $(t_0, t_1, \ldots, t_r)$ have a common prefix (maybe only t). Let c be the last vertex in both paths, and let d be the first vertex of p not in L. Let L_c be the line crossed by L in c. The tile d necessarily is also in L_c. Then $ind_{\{L_c\}}(t, c) = ind_{\{L_c\}}(t, d)$. Thus $|ind_{\mathbf{B}}(t, c) - ind_{\mathbf{B}}(t, d)| \leq 1/2$, which contradicts the hypothesis that p is of length r. $\square$

Lemma 2 *Let T be a tiling of a $2D$-gon, t, t' be two tiles of T having exactly one point in common and having one edge included in a given side of the $2D$-gon. The common point is assumed to belong to the hull of the $2D$-gon. The unique shortest path between t and t' is the fan $(t = f_1, f_2, ..., f_r = t')$, from t to t'. See Figure 11.*

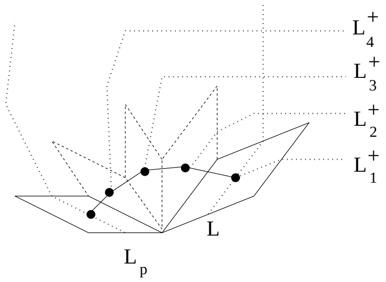


Figure 11. A fan and the unique shortest path between its endpoints (Lemma 2).

Proof: The argument is exactly the same as in Lemma 1, using the set formed by $\{L_i^+ | 1 \leq i \leq p-1\} \cup \{L_p\}$, (we recall that L_p is defined by the edge of t' which is on the hull). $\square$

3.2 Constructing the border

The first step of algorithm to construct a tiling equivalent to a tiling T starting from the graph with origins of T will be to construct its border. The vertices having degree 2 or 3 are obviously part of the border. The point is first to find the vertices of degree 4 which also belong to the border, then to find the succession between vertices along the border. We first show some properties of the graph allowing to make both these at the same time. We then give an algorithm to compute the border of the adjacency graph (Theorem 2), which completes the first step of the construction of T from its graph with origins.

Having shown the shortest path properties, we turn now to the main part of the algorithm. In the following, by "successive tiles" we mean two tiles of the border having degree 2 or 3 and adjacent by a point of the hull of the tiling. This corresponds to two tiles which are endpoints of a same fan. The algorithm we want to build is a greedy one which, starting with two successive tiles of the border, finds a tile successive to one of them. Hence, starting with the origins, this constructs locally the paths between successive tiles of the border, leading to a complete construction of the border.

Let T be a tiling, and t_1, t_2, t_3 be three tiles such that:

- t_1 and t_3 are endpoints of a fan, whose head is a point v (i.e. $v = t_1 \cap t_3 \cap H(T)$)
- t_1 and t_2 are endpoints of a fan, whose head is a point $v' \neq v$ (i.e. $v' = t_1 \cap t_2 \cap H(T)$)

We assume the fan whose head is v' has already been constructed, and we want to construct the fan whose head is v.

Let c_0 be the edge of t_1 on the hull and containing v. c_0 induces a de Bruijn line L, which cuts the tiling into 3 parts: T^+, T^- and L. We assume, without loss of generality, that $t_3 \in T^-$.

Let c' be the side of t_2 on the hull and containing v'. c' induces a de Bruijn line L'.

Let c_p be the side of t_3 on the hull and containing v. c_p induces a de Bruijn line L_p.

Let t_{12} be the tile at the crossing between L and L', if it exists. This tile separates in particular L into L^+ and L^-, L^- being the part from t_1 and t_{12} excluded, and L^+ being the other part of L. t_1 being one endpoint of L, we denote by t_L its other endpoint. Hence L^+ is the part of L from t_{12} to t_L. See Figure 12.

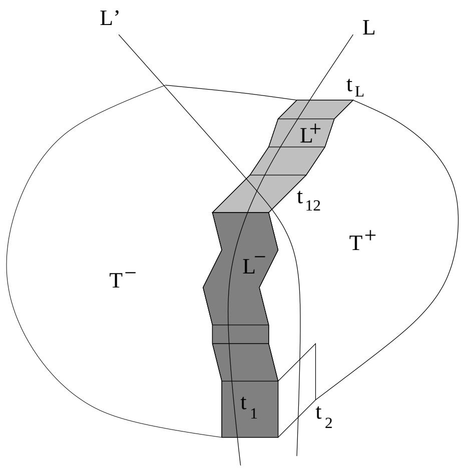


Figure 12. The general case. Light grey : L^+; Dark grey: L^-

Let $t'_1, t'_2, ..., t'_p$ be the tiles of the fan induced by v, in such a way that the path from t_1 to t_3 is $t_1 = t'_1, t'_2, ..., t'_p = t_3$. Let c_i be the edge in common to t'_{i+1} and t'_i. We call L_i the de Bruijn line induced by c_i. See Figure 13.

Notice that c_0 and c_p are the sides on the hull induced by v.

We make the assumption that t_1 and t_3 are not adjacent (otherwise the search for t_3 is obvious).

For further convenience, we call this the *fan search situation*,and we introduce some notations: A_1 is the set of tiles closer to t_1 than to t_2, i.e. $A_1 = \{t \in T, dist(t, t_1) < dist(t, t_2)\}$. In the same way, , $A_2 = \{t \in T, dist(t, t_2) < dist(t, t_1)\}$. $A_=$ will be the set of tiles whose distance to both t_1 and t_2 is the same.

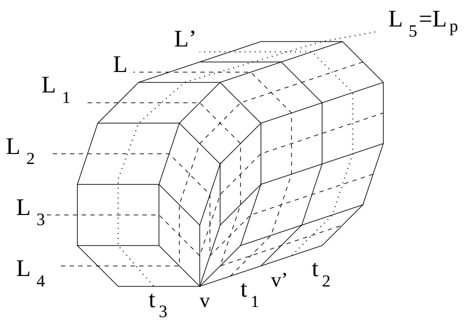


Figure 13. The fan search situation, and the numbering of the lines.

3.2.1 General case: $t_2 \in T^+$

This induces two different cases. Either L, L' and L_p are parallel (which means c_0, c_p and c' have same support), or at least two of them cross each other in the tiling. We start with the last case, which is the general case.

a- c_0, c_p and c' do not all have same support

We assume here that c_0, c_p and c' do not all have same support (at least two of the lines L, L_p and L' cross each other).

Proposition 3 *In the general fan search situation, let $T^+_{hull} = \{t \in T^+, deg(t) < 4\}$. we have:*

(1) $L^- \subseteq A_1$
(2) $L^+ \cup T^+_{hull} \subseteq A_2 \cup A_=$,
(3) if L and L' are parallel (i.e. c_0 and c' have same support), then $L \cup T^- \subseteq A_1$

As a consequence of the second item, if we state: $T^-_{hull} = \{t \in T^- | deg(t) < 4\}$ and $A_{1,hull} = \{t \in A_1 | deg(t) < 4\}$, then we have $A_{1,hull} \subseteq T^-_{hull} \cup \{t_L\}$ where t_L denotes the endpoint of L different from t_1. Moreover, when L and L' are not parallel, $A_{1,hull} \subseteq T^-_{hull}$.

Proof: Let u_i be the i-th tile of L, starting from t_1, and M_i be the line crossing L on u_i. We take $\mathbf{B}_i = \{M_j | 1 \leq j \leq i\}$. We first prove $dist(u_i, t_2) \geq dist(u_i, t_1)$.

Lemma 1 implies that $dist(t_1, u_i) = i-1$. For $1 \leq j < i$, we have $ind_{\{M_j\}}(u_i, t_2) = 1/2$ if $t_2 \in M_j$, and $ind_{\{M_j\}}(u_i, t_2) = 1$ otherwise (since $u_i \notin M_j$ for $i \neq j$). Moreover, we have: $ind_{\{M_i\}}(u_i, t_2) = 0$ if $t_2 \in M_i$, $ind_{\{M_i\}}(u_i, t_2) = 1/2$ otherwise. Since t_2 is crossed by two de Bruijn lines, at most two of the preceding values are minimal. This induces that $ind_{\mathbf{B}_i}(u_i, t_2) \geq (i - 1/2) - 2 \times 1/2$, thus $dist(u_i, t_2) \geq i - 1$. Hence $dist(u_i, t_2) \geq dist(u_i, t_1)$.

(1) Let $u_i \in L^-$. Assume $dist(t_2, u_i) = i-1$, and let $(u_i = r_0, r_1, r_2, ..., r_{i-1}, r_i = t_2$ be a $(i-1)$-long path from u_i to t_2. We necessarily have: $ind_{\mathbf{B}_i}(u_i, r_j) = j$. We easily prove by induction that, for $0 \leq j \leq i$, $r_j = u_{i-j}$: the initialization of the induction is obvious, and assuming $r_j = u_{i-j}$, since $ind_{\mathbf{B}}(u_i, r_{j+1}) = ind_{\mathbf{B}}(u_i, r_j) + 1$, one has to leave M_j, i.e. $r_{j+1} \in L$, thus $r_{j+1} = u_{i-j-1}$. This induces $t_2 = t_1$, which contradicts the assumptions. Hence $dist(u_i, t_2) > dist(u_i, t_1)$.

(2) Let $t \in L^+$. The unique shortest path from t to t_1 is following L. Moreover, the path from t to t_2 consisting in following L^+ until t_{12}, then L' until t_2, has length $dist(t, t_1)$, because $t_{12} \in A_=$ (since each line crosses L^- if and only if it crosses L'^-). Then $L^+ \subseteq A_= \cup A_2$.
Let now $t \in T^+_{hull}$.
   (a) either the shortest path from t to t_1 contains at least one tile of L^+ (in this case a suffix of this path is a part of L), and one can construct a path of same length leading from t to t_2 by replacing the part from t_{12} to t_1 by the path from t_{12} to t_2 as above. Thus $t \in A_= \cup A_2$ (there may exist a shorter path from t to t_2).
   (b) in the other case, the shortest path from t to t_1 necessarily crosses L' on a tile $t_{L'}$ of L'^- (the part of L' from t_2 to t_{12}, with t_{12} excluded). The argument used in 1. just above induces $t_{L'} \in A_2$. Hence $t \in A_2$.
This leads to the result.

(3) If L and L' are parallel, $L = L^-$, and 1. induces the result for L. For T^-, the case is symmetrical to 2.(b). The same argument leads to the conclusion (just exchange L and L', and T^+ and T^-). □

Lemma 4 *Let $A_{1,hull} = \{t \in A_1, deg(t) < 4\}$. We have :*

(1) $\forall t \in A_{1,hull}$, $dist(t, t_1) \geq dist(t_3, t_1)$.
(2) If $dist(t, t_1) = dist(t_3, t_1)$ then $t \in L_p$.

Proof: Let $t \in A_{1,hull}$. From the preceding lemma, either $t \in T^-_{hull}$, or $t = t_L$.

If $t \in T^-_{hull}$:

- $ind_{L_1^+}(t, t_1) = 1/2$
- $ind_{L_i^+}(t, t_1) = 1$ for $2 \leq i \leq p - 1$
- $ind_{L_p}(t, t_1) = 1/2$ if $t \in L_p$, 1 otherwise

Then, if $\mathbf{B}$ is the union of the preceding lines, we have $ind_{\mathbf{B}}(t, t_1) \geq p - 3 + 1 + 1/2 + 1/2 = p - 1$, i.e. $dist(t, t_1) \geq dist(t_3, t_1)$. Moreover, the equality can be obtained only for $ind_{L_p}(t, t_1) = 1/2$, i.e. $t \in L_p$.

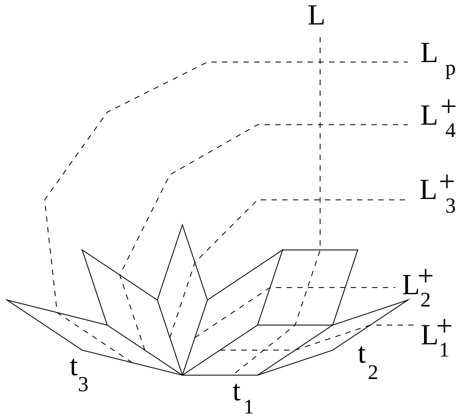


Figure 14. The lines that have to be crossed to reach the hull, in the fan search situation.

If $t = t_L$, L and L' are parallel, otherwise $t \in A_2 \cup A_=$. Then L and L_p cross each other (we are not in the case where L, L' and L_p are parallel), and $ind_{L_p}(t, t_1) \geq 1/2$. This leads directly to the same result as the previous case.$\square$

Lemma 5 *Let $t \in A_{1,hull}$ such that $dist(t, t_1) = dist(t_3, t_1)$. Let ($t_3 = r_0$, r_1, …, $r_k = t$) be the geodesic from t_3 to t. For each integer i such that $0 \leq i \leq k$, $r_i \in A_{1,hull}$.*

Proof: Let $t \in L_p$ such that $dist(t, t_1) = dist(t_3, t_1)$, and let ($t_1 = s_0$, s_1, …, $s_{p-1} = t$) be a geodesic from t_1 to t.

This geodesic meets successively L_1^+, L_2^+, …, L_{p-1}^+, L_p. Since $dist(t, t_1) = p-1$, no other line is crossed by the geodesic, thus s_0 is necessarily on L_1^+, s_1 is on L_2^+, … and s_{p-1} is on L_p.

Let j denote the index of the last s_i of the fan. Assume that $j < p-1$. From the condition above, s_{j+1} is on L_{j+2}^+, thus s_{j+1} necessarily is in L_j^-, as s_j. For the same reason (if $j+1 < p-1$), s_{j+2} is also in in L_j^-, and so on until s_{p-1} is reached.

The set of de Bruijn lines crossing the geodesic (i. e. L_j^-) from s_j to t is $\{L_{j+1}, L_{j+2}, \ldots, L_p\}$. Moreover these lines are exactly those which reach the hull on the part of T^- between t and t_3.

Thus, starting from the fan, L_{j+1}^- successively crosses L_{j+2},L_{j+3},… and L_p, and no other line is crossed by L_{j+1}^-. One can repeat the argument: starting from the fan, L_{j+2}^- successively crosses L_{j+3}, L_{j+4}… and L_p, and no other line is crossed by L_{j+2}^-.

On the other side, for $j \leq j' \leq p-1$, from $s_{j'}$ to the fan, the line $L_{j'+1}^+$ successively meets $L_j, L_{j+1}, \ldots, L_{j'}$ and no other line is crossed by $L_{j'+1}^+$ between

$s_{j'}$ and the fan (with the convention $L_p = L_p^+$ for $j' = p-1$).

This induces the result: the line L_p follows the hull between t_3 and t and, for each integer i such that $0 \leq i \leq k$, one easily constructs a path of length $p-1$ from t_1 to r_i. $\square$

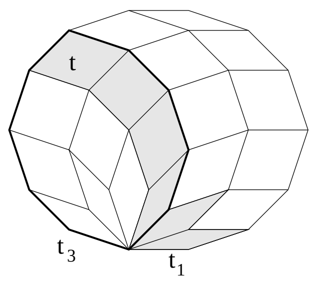


Figure 15. Proof of lemma 5: the unitary zonotope (bold), gray is the geodesic from t_1 to t. The fan from t_1 to t_3 enforces the tiling of the whole unitary zonotope, which gives the result.

Proposition 6 *Let X_3 denote the subset of $A_{1,hull}$ formed by tiles t such that $dist(t, t_1) = dist(t_3, t_1)$. The tile t_3 is the unique element of X_3 whose degree is lower than the degree of all the other tiles of X_3.*

Proof: This is obvious, from the previous lemma once it has been seen that t_{L_p}, the other endpoint of L_p, cannot be in X_3: if L_p and L cross each other, then L_p is not in A_1, from lemma 1; otherwise, L_p and L' cross each other, which implies that $dist(t_{L_p}, t_1) > dist(t_3, t_1)$, using the set formed by the lines L_i^+ for $1 \leq i \leq p-1$, and L'. $\square$

b- c_0, c_p and c' have same support

We first prove a particular property of this case, and then show some particular situations leading to a further study. All takes place in the fan search situation, with c_0, c_p and c' having same support.

Lemma 7 *Let $A_{1,hull} = \{t \in A_1, deg(t) < 4\}$. Then $\forall t \in A_{1,hull}$, $dist(t, t_1) \geq dist(t_3, t_1) - 1$. Moreover, if there exists t such that $dist(t, t_1) = dist(t_3, t_1) - 1$, then $t = t_L$, t_L being the endpoint of L opposite to t_1.*

Proof: We can use the arguments of the similar lemma of the previous case, except when $t = t_L$: in this case only, it is possible to have $ind_{L_{p-1}^+}(t, t_1) = 1/2$. Thus we obtain $ind_{\mathbf{B}}(t, t_1) \geq p - 3 + 1/2 + 1/2$, which gives $dist(t, t_1) \geq dist(t_3, t_1) - 1$ and the equality can be obtained only for t_L. $\square$

Two particular cases may occur, which we study now, keeping the same notations as for the preceding lemma:

(1) $dist(t_L, t_1) = dist(t_3, t_1) - 1$.
This case corresponds to the situation described Figure 16: since L is crossed by each half-line L_i^+ on a tile u_i, The geodesic from t_1 to t_L is necessarily $(u_1, u_2, ..., u_{p-1})$, ($t_1 = u_1$ and $t_L = u_{p-1}$). Since L and L_p are parallel, L_p also contains $p-1$ tiles. Moreover, starting from t_3, L_p successively crosses $L_{p-1}, L_{p-2}....L_1$. Starting from t_2, L' successively crosses $L_1, L_2....L_{p-1}$.

An important point for the following is that u_2 is adjacent to a tile of A_2. This fact allows to discover that we are in this case.

Let X_3 denote the set $\{t \in A_1,\ \ dist(t, t_1) = dist(t_3, t_1)\}$ and X_{min} denote the set of elements of X_3 of minimal degree. We have: $t_3 \in X_{min}$, $card(X_{min}) = 2$, and the element t_4 of X_{min} different from t_3 is the other endpoint of L_p.

Notice that $dist(t_4, t_L) = 2$, and $dist(t_3, t_L) = p - 1$, which is at least 3, since we have dropped the particular cases when the border is not a cycle.

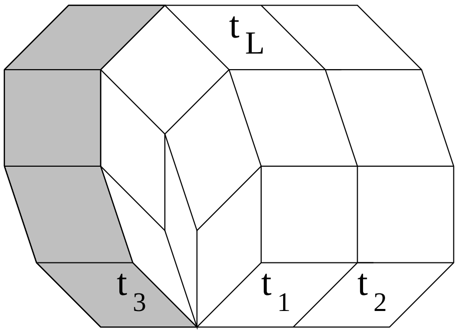


Figure 16. The particular case in which t_L is closer to t_1 than t_3, and all the tiles of L_p (in gray) are at same distance of t_1.

(2) $dist(t_L, t_1) = dist(t_3, t_1)$.
In this case, from the previous situation, a de Bruijn line L'' non parallel to L, L' and L_p is added to the tiling, since L contains all the tiles u_i as above and one other tile. This situation can be seen in Figure 17. The space between L and L' only contains some tiles of L'', and those tiles are in $A_=$.

With the notations above $card(X_{min}) \leq 2$, and, if $card(X_{min}) = 2$, the element of X_{min} different from t_3 is t_L.

One can notice that the tile of L adjacent to t_1 is adjacent to a tile of $A_2 \cup A_=$, and that the tile of the fan adjacent to t_1 has not the same property, since it is in T^-.

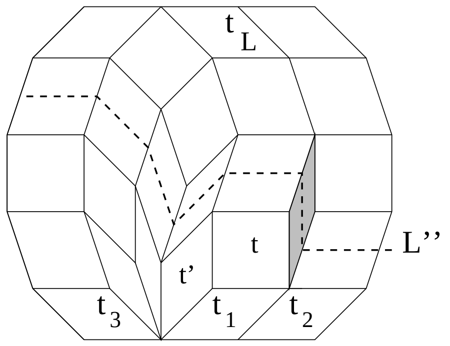


Figure 17. Particular case with a line crossing L. Notice that the tile adjacent to t (in gray) is in $A_= \cup A_2$, while none of the tiles adjacent to t' is.

3.2.2 Particular case: $t_2 \in L$

Now we have seen how to find the successor of a tile on the border in the case $t_2 \in T^+$, we have to study the particular case $t_2 \in L$. We proceed as in the general case. The difference comes from the criterion allowing to exclude tiles in T^+

Lemma 8 *Considering the fan search situation, if $t_2 \in L$, then:*

(1) let t_x be a tile of $T^+ \cup L \backslash \{t_1\}$. Then $t_x \in A_2$. More precisely, each geodesic from t_1 to t_x contains t_2
(2) $t_3 \in A_1 \cup A_=$

Proof:

(1) If $t_x \in L$, then the geodesic from t_x to t_1 is included in L. Since the only tile of L adjacent to t_1 is t_2, $t_x \in A_2$. Hence $L \backslash \{t_1\} \subseteq A_2$. If $t_x \in T^+$, consider a shortest path from t_1 to t_x. Assume this path does not start by $(t_1, t_2, ...)$. Then it starts by $(t_1, t', ..)$, with $t' \in T^-$. Thus the path contains a tile t'' (different from t_1) which is in L, since $t_x \in T^+$. By Lemma 1, the unique shortest path from t_1 to t'' follows L, which contradicts the assumption, and $t_x \in A_2$.
(2) It is clear that $ind_L(t_2, t_3) = ind_{L_p}(t_2, t_3) = 1/2$. Moreover, for $3 \leq i \leq p-1$, $ind_{L_i^+}(t_2, t_3) = 1$, and $ind_{L_2^+}(t_2, t_3) \geq 1/2$. Thus, considering the set $\mathbf{B}$ of all these lines, we obtain: $ind_{\mathbf{B}}(t_2, t_3) \geq p - 3 + 3 \times 1/2$. This gives $dist(t_2, t_3) \geq p - 1$. $\square$

Lemma 9 *Let $A_{1,=,hull} = \{t \in A_1 \cup A_=, deg(t) < 4\}$. Then :*

(1) $\forall t \in A_{1,=,hull}$, $dist(t, t_1) \geq dist(t_3, t_1)$.
(2) If $dist(t, t_1) = dist(t_3, t_1)$ then $t \in L_p$.

We do not give the proof since it is very similar to lemma 4.

This lemma induces in particular that $card(X_{min}) \leq 2$, where X_{min} is defined in an analogous way as in the previous case. Since the elements such that $dist(t, t_1) = dist(t_3, t_1)$ are in L_p, the two possible elements of X_{min} are t_3 and t_{L_p}.

If $card(X_{min}) = 2$, we are in a case similar to Figure 16, and there is a unitary zonotope between L and L_p. Lemma 9 underlines the fact that $L \setminus \{t_1\} \in A_2$. By construction $dist(t_{L_p}, t_L) = 2$, i.e. $dist(t_{L_p}, A_2) = 2$. Moreover, with the same notations as previously, taking $\mathbf{B}$ as the set $\{L_i^- | 1 \leq i \leq p-1\}$ together with L_p, we have $ind_{\mathbf{B}}(t_3, t_L) \geq p - 1 + 1/2$, since $ind_{L_p}(t_3, t_L) = 1/2$, $\forall i, \;\; ind_{L_i^-}(t_3, t_L) = 1$. Moreover, $p - 1 \geq 2$ (otherwise there is one tile in the fan, which is adjacent to t_1, t_3, t_L and t_{L_p}, and this particular case has been excluded). Then $dist(t_3, A_2) \geq 3$, since all other tiles of A_2 are further from t_3 than L (see lemma 8).

3.2.3 Algorithm for constructing the border

We have now all the preliminary results necessary to write an algorithm which constructs the border of the adjacency graph of a $2D$-gon tiling (Algorithm 2).

In the fan search situation, we make a breadth first search in A_1, starting from t_1, and select the tiles closest to t_1 with degree < 4. This gives a set of tiles, one of which is t_3 in most of the cases; we call these *candidates*.

In the general case, either there is only one candidate, and it is t_3, or there are several, and the one with minimal degree is t_3.

The preceding study enlightens particular cases:

- There may exist a unique candidate which is not t_3 (see lemma 7). This is the case of Figure 16, and the candidate is t_L. Then there exists a tile neighbor to both t_2 and the successor of t_1 on the path from t_1 to t_L. We can perform another breadth first search in $A_1 \backslash L$, since L is the path from t_1 to t_L.
- There may exist two candidates with minimal degree, one of them is t_3. Two cases occur:
  - · if the other candidate is t_{L_p}, then we are in the case of Figure 16 (after having performed a breadth first search in $A_1 \backslash L$). Thus there is a 2-long path from t_{L_p} to t_L, which allows to find t_3. This can also occur in the particular case of lemma 9, and the wrong candidate is a distance-2 neighbor of a tile of A_2.
  - · in the other case, the candidate is t_L, and we are in the case of Figure 17.

Then one can find t_3 in the search tree induced by the breadth first search, by starting at t_1 and selecting at each step the successor having degree 3 (i.e. having 2 successors). One thus obtains two vertices, the one having degree < 4 is t_3.

These remarks lead to the following procedure:

(1) if t_3 and t_1 are adjacent, then the result is obvious
(2) if t_2 and t_1 are adjacent, and t_1 has degree 2, then $t_2 \in L$. We make a breadth first search to find $X_{min} = \{t \in A_{1,=}, deg(t) < 4, dist(t, t_1) \text{ minimal}\}$. We have the alternatives below:
   (a) X_{min} is reduced to one single element, and this one is t_3,
   (b) X_{min} has two elements. Thus, L and L_p are parallel. and lemma 7 induces that there exists a unique t in X_{min} such that $dist(t, A_2) \geq 3$. This one is t_3.
(3) in other cases, $t_2 \in T^+$, we make a breadth first search to find $X_{min} = \{t \in A_1, deg(t) < 4, dist(t, t_1) \text{ minimal}\}$.
   We have the following alternatives:
   (a) X_{min} has one element t. We make a test to know whether this element is t_3. Let t_{test} be the first tile in the shortest path from t_1 to t found previously.
      (i) If $\exists t' \neq t_1$ such that t' is adjacent to t_2 and t_{test},then $t \neq t_3$: we are in the case of Figure 17. We make a new breadth first search to find $X_{min2} = \{t \in A_1 \setminus L, deg(t) < 4,\ dist(t, t_1) \text{ minimal}\}$. Then X_{min2} has two elements. One of them is at distance 2 of t. The other one is t_3.
      (ii) otherwise, $t = t_3$.
   (b) X_{min} has two elements. Then $X_{min} = \{t_3, t_L\}$, and L, L' and L_p are parallel. We make a test to recognize t_3. Let t be a tile of X_{min} and t_{test} be the first tile in the shortest path from t_1 to t found previously. If there exists one tile of $A_= \cup A_2$ adjacent to t_{test}, then $t = t_L$, otherwise $t = t_3$.

Theorem 2 *Given the adjacency graph of a $2D$-gon tiling, Algorithm 2 computes its border in time $O(m \cdot n)$ where n is the number of vertices of the graph, and m is the sum of the multiplicities defining the $2D$-gon.*

Proof: The basic complexity of this greedy algorithm is $O(n \cdot m)$, where n is the number of vertices and m the sum of the multiplicities, because it is just visiting all the vertices with degree at most 3, and builds a breadth first search in the rest of the graph, where every vertex has degree at most 4. Then, starting with one origin, it only has to follow the distance 1 vertices until there is none. The treatment of the particular cases can also be done in $O(n \cdot m)$. This leads to a global time complexity $O(n \cdot m)$. □

Algorithm 2: Construction of the border of the adjacency graph of a $2D$-gon tiling.

Input: $G = (V, E, v_1, v_2)$ the graph with origins of a $2D$-gon tiling.

Output: The border of G, $\mathcal{B}(G)$, as an ordered list of vertices.

```
begin
  // we assume the function BFS(X,t) computes the set of
  // closest vertices to t in X with degree < 4 and minimal.
  Let B be the set of vertices in V of degree< 4;
  t_1 ← v_2, t_2 ← v_1;
  𝓑 ← shortestpath(t_2, t_1) excluding t_1;
  repeat
    if ∃t ∈ B, t ≠ t_2, (t_1, t) ∈ E then
      t_3 ← t; //case 1. t_1 and t_3 are adjacent
    else
      if (t_1, t_2) ∈ E and deg(t_1) = 2 then
        X_min = BFS(A_1 ∪ A_=, t_1); //case 2. t_2 ∈ L
        if card(X_min) = 1 then
          t_3 ← X_min; //case 2(a)
        else
          Let t ∈ X_min, dist(t, A_2) ≥ 3; // case 2(b)
          t_3 ← t;
      else
        X_min = BFS(A_1, t_1); //case 3
        if card(X_min) = 1 then
          let t_test be the first tile on the path (t_1, X_min); //case 3(a)
          if ∃t' ≠ t_1 ∈ V, (t_test, t') ∈ E, (t_1, t_2) ∈ E and (t_2, t') ∈ E
          then
            X_min2 = BFS(A_1 \ path(t_1, X_min), t_1); // case3(a)i.
            t_3 ← t, t ∈ X_min2, dist(t, X_min) ≥ 3;
          else
            t_3 ← X_min; //case 3(a)ii.
        else
          // case 3(b);
          let t ∈ X_min, and t_test the first tile on the shortest path
          (t_1, t);
          let t'' be the other vertex in X_min;
          if ∃t ∈ V, (t, t_test) ∈ E, t ∈ A_= ∪ A_2 then t_3 = t'' else
          t_3 = t;
    Add the shortest path from t_1 to t_3 (excluding t_3) at the end of 𝓑 ;
    t_2 ← t_1, t_1 ← t_3;
  until t_3 = v_1;
  Return(𝓑);
end
```

We will now show that, when one knows the border of the graph with origins of a $2D$-gon tiling T, then one can construct the de Bruijn lines of T by computing shortest paths in the graph. Indeed, when the border is constructed, one knows the succession of tiles along the hull of the tiling. Then, one can find the j-th tile for each $j \in 1, ..., m$, m being the sum of the multiplicities (with the convention that a tile with two edges on the hull is simultaneously the j-th and the $(j+1)$-st of the sequence, the first tile is the origin v_1). By the construction of the tiling, one can easily check that the j-th and the $(m+j)$-th tile correspond to endpoints of lines of a same family .

The two following lemmas show that taking one tile on one side of the tiling and computing the shortest paths to all the tiles on the opposite side of the tiling, one finds the de Bruijn line as being the one with minimal length.

Notice that m can be easily computed from the adjacency graph: we have $2m = 2T_2 + T_3$, where T_2 denotes the number of tiles of degree 2 and T_3 denotes the number of tiles of degree 3.

Lemma 10 *Let T be a tiling of a $2D$-gon, t and t' be two tiles of T, with t on the side i and t' on the side $m+i$ of the $2D$-gon. This induces that t and t' are endpoints of lines of the i-th family. Let L_t (respectively $L_{t'}$) be the line of the i-th family containing t (resp. t'). Assume r is the number of lines of the i-th family not placed between L_t and $L_{t'}$. Then $dist(t, t') = m - r - 2$ (where m is the sum of the multiplicities). See Figure 18.*

Proof: We consider the line value associated with all the de Bruijn lines except the r lines of the i-th family which are not between L_t and $L_{t'}$. Each of these lines has to be crossed, except those containing t of t'. Then the line value is $(m-r)-4\times 1/2 = m-r-2$ ($1/2$ is removed each time a line contains t or t'). Thus the length of the path from t to t' is at least $m - r - 2$.

Let now L' be the de Bruijn line crossing L_t at t. Consider the path consisting in following L' until $L_{t'}$ is reached, then following $L_{t'}$ until t' is reached. Each step consists in leaving one line and reaching another, so it increases the line value by 1. Hence this path is $m - r - 2$ steps long. $\square$

Lemma 11 *Let T be a tiling, t, t' be two tiles, which are endpoints of a de Bruijn line L, and t'' be a tile such that there exists a fan from t' to t''. We have $dist(t, t') \leq dist(t, t'')$. See Figure 19.*

Proof: Let $(t = t_0, t_1, \ldots, t_r = t')$ be the path induced by L starting in t. For each integer i such that $1 \leq i < r$, let L_i denote the de Bruijn line crossing L in t_i, and b_i the value 1 if $i \neq 0$, and $b_0 = 1/2$. Remark that $ind_{L_i}(t, t'') = b_i$

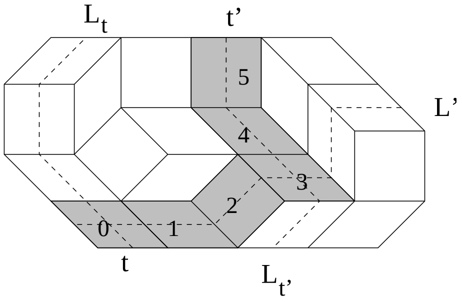


Figure 18. The general case, and the values associated with one path from t to t'.

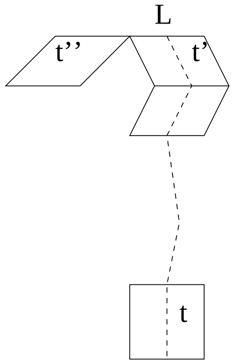


Figure 19. The positions of the tiles for lemma 11.

if t'' is not on L_i and $ind_{L_i}(t, t'') = b_i - 1/2$ if t'' is on L_i. Thus, taking the family $\mathbf{B}$ of lines L_i, we have: $ind_{\mathbf{B}}(t, t'') \geq \sum_{i=0}^{r} b_r - 1$, since t'' is on at most two lines L_i. Therefore, $ind_{\mathbf{B}}(t, t'') \geq r - 1/2$. Moreover $dist(t, t') = r$. Thus $dist(t, t'') \geq ind_{\mathbf{B}}(t, t'') \geq dist(t, t') - 1/2$, which gives the result, since $dist(t, t'')$ is an integer. □

We then have a criterion to find the endpoints of each de Bruijn line. It is clear that taking one tile t_1 and the tile t_2 placed at the opposite of t_1 in the border (i.e. t_1 is at the α-th position, and t_2 is at the $(\alpha + m)$-th, m being the sum of multiplicities), one obtains two endpoints of lines of a same family. Moreover we assume that the $(\alpha - 1)$-th position is not in the same side as the α-th position.

We consider the tile t_3, placed after t_2 in the hull. If t_3 is in the same family as t_2, then $dist(t_1, t_3) = dist(t_1, t_2) - 1$. In the other case, then $dist(t_1, t_3) \geq dist(t_1, t_2)$. This gives an algorithmic criterion to find the de Bruijn lines.

Precisely, the algorithm computes the de Bruijn lines as follows: it first computes the distance $dist(t_1, t_2)$. Then, it moves from t_2 to its successor tile in the hull (following the border along a fan), say t_3, and computes $dist(t_1, t_3)$. If $dist(t_1, t_3) = dist(t_1, t_2) - 1$, we are in the same side (lemma 10). Then we go on until $dist(t_1, t_j) \geq dist(t_1, t_{j-1})$, where t_j is the $(j-2)$-th successor of

t_2 in the border. At this point, we know that $t_2, ..., t_{j-1}$ and t_1 are the end-points of lines of the same family (from lemma 11). Then the de Bruijn lines of this family are the shortest paths between successively $(t_1, t_{j-1}), (t'_1, t_{j-2}), ...,$ and$(t_1^{(j-2)}, t_2)$, where the $t_1^{(k)}$ is the k-th successor of t_1 in the hull.

This allows to compute an entire de Bruijn family, and processing for each family, we obtain the whole tiling. This remarks lead directly to the following theorem.

Theorem 3 *Given the graph with origins of a $2D$-gon tiling T and its border, Algorithm 3 computes the de Bruijn graph of T in time $O(n \cdot m)$ where n is the number of vertices of the graph and m is the sum of all the multiplicities which define the $2D$-gon.*

Starting from the graph with origins of a $2D$-gon tiling T, it is now clear that one can construct a tiling equivalent to T by computing the border of the graph with Algorithm 2, then compute the de Bruijn graph with Algorithm 3, and finally obtain the tiling using Algorithm 1. Therefore, we can finally combine Theorems 1, 2 and 3 to obtain:

Theorem 4 *Given the graph with origins of a $2D$-gon tiling T, there is an algorithm which constructs a $2D$-gon tiling equivalent to T in time $O(m \cdot n)$, where n is its number of tiles, and m the sum of the multiplicities of the vectors used to define the $2D$-gon.*

This result not only gives an efficient and simple way to encode and manipulate 2D-gon tilings; it also clarifies the relation that exists between the adjacency graph of a tiling and the tiling itself. In particular, it proves that, despite the fact that there is no one-to-one correspondence between adjacency graphs and tilings, the adjacency graph contains almost all the information on the tiling.

Algorithm 3: Construction of the de Bruijn graph of a $2D$-gon tiling from its graph with origins.

Input: $G = (V, E, v_1, v_2)$ the graph with origins of a $2D$-gon tiling T and $\mathcal{B}(G)$ its border.

Output: $G' = (V, E, \lambda, \nu)$, the de Bruijn graph of T.

```
begin
    i ← 1 ; //computation of m = size(B)/2
    foreach x ∈ B(G) do
        if deg(x) < 4 then
            B[i] ← x, i ← i + 1;
            if deg(x) = 2 then
                B[i] ← x, i ← i + 1;
    α ← 1 ;
    repeat
        β ← α + ⌊size(B)/2⌋ ; //α andβ are the indices of the endpoints
        dist[0] ← distance(B[α], B[β]); //dist[i]:length of the i-th path currentpath ← shortest path(B[α], B[β]);
        endfamily ← false; //currentpath contains the last computed path l ← 0; // loop for determining the size l of a de Bruijn family
        while endfamily=false and β + l < size(B) do
            l ← l + 1;
            tmp ← shortest path(B[α], B[β + l]);
            dist[l] ← distance(B[α], B[β + l]);
            if dist[l] < dist[l − 1] then
                currentpath ← tmp
            else
                endfamily ← true
        //computation of the lines of the current family
        foreach vertex v of currentpath do
            λ(v) ← λ(v) ∪ {α};
        l ← l − 2; α ← α + 1;
        while l ≥ 0 do
            build the shortest path between B[α] and B[β + l];
            foreach vertex v of the path do
                λ(v) ← λ(v) ∪ {α};
            l ← l − 1; α ← α + 1;
    until α > ⌊size(B)/2⌋;
    Return (V, E, λ, v1);
end
```

4 An application: random tilings

Tilings of $2D$-gons are an important model of quasicrystals in physics. In this context, it is very important to be able to sample random tilings, which helps the study of the entropy of the quasicrystal [WMDB02]. The sampling uses the key notion of *flip*: given a $2D$-gon tiling, one may rearrange locally three tiles (which form an hexagon) in order to obtain a new tiling of the same $2D$-gon (see Figure 20).

This enables the random generation of tilings of a $2D$-gon: it is shown in [Eln97,Ken93] that one can obtain all the tilings of a $2D$-gon from a given one by iterating the flip operation. When one wants to obtain a random tiling, one then has to choose a particular tiling and then iterate the flip operation *until the obtained tiling can be considered as random.* This notion of *when* one can stop the process is central when one wants to sample random tilings *with the uniform distribution.* It is possible to sample perfectly random tilings of hexagons because of the distributive lattice structure of the set of all the tilings [Pro98]. This technique can no longer be used for octagon, but a recent study explains how long the process has to be continued in order to be as close as one may want of the uniform distribution [Des01]. For the other $2D$-gons, *i.e.* when $D > 4$, there are no known results [WMDB02].

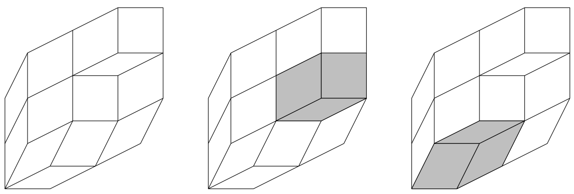


Figure 20. A tiling of an octagon ($D = 4$) (left) and two other tilings of the same octagon obtained from the first one by a flip (the shaded tiles are the ones which moved during the flip).

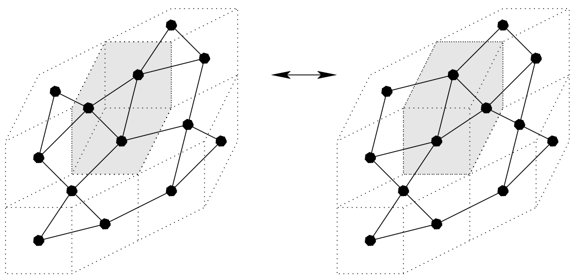


Figure 21. A flip on a $2D$-gon, and on its graph.

Therefore, when one wants to sample a random tiling of a given $2D$-gon P, the only solution is to construct a particular tiling of P and iterate the flip operation. To achieve this, one can use the graphs encodings we proposed above: the flip operation can be encoded on the graph, as shown in Figure 21. The vertices which correspond to the tiles to flip form a triangle in the graph, and conversely, all the triangles in the graph correspond to a possible flip in the tiling. Moreover, the transformation on the graph is a local rearrangement of vertices. The flip just exchanges the links of edges having the same label. Indeed, it just corresponds to exchanging the order between tiles within each de Bruijn line. Hence, considering a line, the tiles are exchanged during a flip, which corresponds to linking each tile with the preceding neighbor of the other tile. Notice however that this operation needs the labels, and cannot be performed on the adjacency graph, which is unlabeled. This makes it possible to implement the flip operation very efficiently and so to iterate it a very high number of times. We show in Figure 22 a random tilings of a decagon obtained this way.

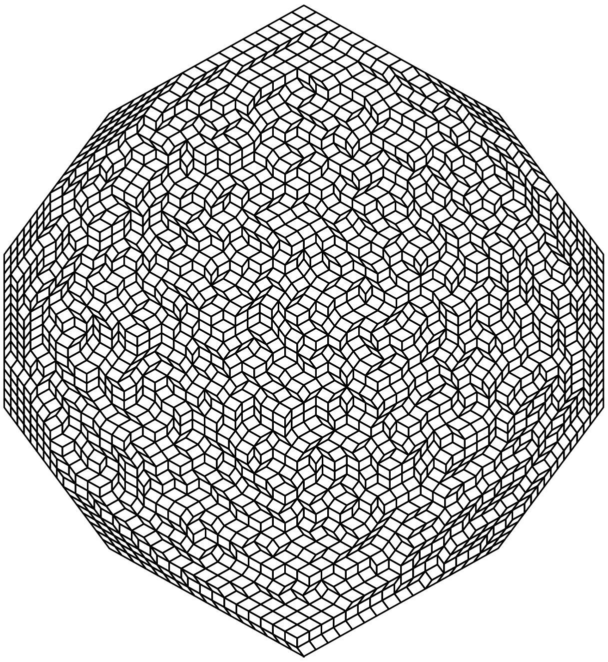


Figure 22. a random tiling of a $5 \to 2$ tiling with side size 16, obtained after 100 million flips.

5 Perspectives

The algorithmic study of tilings of $2D$-gons is only at its beginning, and many open problems still exist. We cited the problem of knowing how many flips have to be done in order to obtain random tilings with a distribution *close* to the uniform distribution. Another important area is the generation of all the tilings of a $2D$-gon, and their enumeration. The encodings with graphs may be used to study these problems. For example, one may obtain a characterization of which graphs are the graphs associated with a $2D$-gon tiling: these graphs are planar, the degree of each vertex is at most four, and they may have many other properties which could help in generating and counting them.

Moreover, $2D$-gons are a special class (the dimension 2 case) of a very important class of objects, namely *zonotopes* [Zie95]. These objects can be viewed as generalizations of $2D$-gons in higher dimensions, and they play an important role in combinatorics and physics. They are also strongly related to oriented matroid theory [RGZ94]. Many studies already deal with these objects, but their algorithmic manipulation is still a problem, while it would help a lot in verifying conjectures, compute special tilings, and compute some statistics over them. The results presented here may be extended to this more general case, leading to other classes of graphs with interesting properties. Notice however that this generalization is not obvious, since our proofs deeply use properties related to the dimension 2. It is well known in zonotopes theory that there is a gap of complexity between 2-dimensional zonotopes ($2D$-gons) and 3-dimensional ones [Zie95].

Acknowledgments

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