On the tiling system recognizability of various classes of convex polyominoes

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Abstract

We consider some problems concerning two relevant classes of two-dimensional languages, i.e. the *tiling recognizable languages*, and the *local languages*, recently introduced by Giammarresi and Restivo and already extensively studied. We show that various classes of convex and column-convex polyominoes can be naturally represented as two-dimensional words of tiling recognizable languages. Moreover we investigate the nature of the generating function of a tiling recognizable language, providing evidence that such a generating function need not be D-finite.

1 Introduction

In this paper we consider the problem of representing, in terms of *two-dimensional languages*, various classes of *polyominoes*.

Two-dimensional languages are currently an active research field in the theory of formal languages, and several models to recognize or generate two-dimensional objects have been proposed in the literature. Most of them were born with the aim of inheriting the main properties of formal uni-dimensional languages (or *string languages*). Thus, for instance, there were introduced: (regular) operations for two-dimensional languages, finite-state machines, grammars, and recently some attempts of developing a hierarchy of two-dimensional languages have been made. Here we will use the *tiling system recognizability*, introduced in [GRST]. This definition emphasizes conceptual simplicity and a close relation to conventional

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finite automata. Moreover this notion of recognizability is equivalent to another one defined by means of a particular kind of cellular automata called $2-OTA$ [IT77], [IT92]. For the main results and properties of such a topic we refer the reader to [GR, IT90].

The other main ingredient of our work is the class of polyominoes. Polyominoes are well known combinatorial objects [Go], and are related to many different problems, such as: tiling [BN, Po], games [Ga], and enumeration [BM]. These objects are not only interesting for computer scientists, but also remarkable in the study of lattice models in physics and chemistry (for example, in models of polymers, of cell-growth, of percolation... [G, PS]).

In the recent literature, various kinds of problems, including those previously mentioned, on different classes of polyominoes, have been studied by means of a coding of the class in terms of a string language, see for instance [BN], [DV]. Here we aim at providing an analogous representation of polyominoes, but in terms of two-dimensional languages, which turn out to be more capable than string languages. At the same time, such a coding gives us some interesting information about the combinatorial properties of two-dimensional languages, in particular concerning the nature of their generating functions.

1.1 Local languages and tiling systems: basic definitions

In this section we briefly recall the definitions of *local picture languages* and *tiling systems*, and the main properties which will be useful to comprehend the rest of the paper. For more details on two-dimensional languages we refer to [GR].

Given a finite alphabet Σ, we define *picture* of size (m, n) over Σ, a two dimensional rectangular array of elements of Σ having m rows and n columns.

Following the notation introduced in [GR], we surround a (m, n) picture p with a special symbol, indicated by $\#$, not contained in Σ, so that we obtain a new picture $\hat{p}$ of size $(m+2, n+2)$ (see Fig. 1). This *boundary symbol* results extremely useful in the general framework of two-dimensional languages, when scanning strategies for pictures are requested, while in our contest it will be used to guarantee the rectangular shape of each picture.

Moreover, for any $h \leq m, k \leq n$, we denote by $B_{h,k}(p)$ the set of all blocks (or sub-pictures) of p of size (h, k). A *tile* is a sub-picture of size $(2, 2)$.

$$p = \begin{array}{|c|c|c|c|c|}\hline 1 & 4 & 4 & 4 & 4 \\ \hline 1 & 2 & 1 & 1 & 4 \\ \hline 1 & 1 & 2 & 2 & 2 \\ \hline 1 & 1 & 3 & 2 & 2 \\ \hline \end{array} \qquad \hat{p} = \begin{array}{|c|c|c|c|c|c|c|}\hline \# & \# & \# & \# & \# & \# & \# \\ \hline \# & 1 & 4 & 4 & 4 & 4 & \# \\ \hline \# & 1 & 2 & 1 & 1 & 4 & \# \\ \hline \# & 1 & 1 & 2 & 2 & 2 & \# \\ \hline \# & 1 & 1 & 3 & 2 & 2 & \# \\ \hline \# & \# & \# & \# & \# & \# & \# \\ \hline \end{array}$$

Figure 1: A picture p in the alphabet $\Sigma = \{1, 2, 3, 4\}$, and the picture $\hat{p}$ obtained by surrounding p with the symbol $\#$.

Definition 1.1 *Let Σ be a finite alphabet and Σ^{**} the set of all possible pictures over Γ. A two dimensional language $L \subseteq \Sigma^{**}$ is local if there exists a finite set θ of tiles over the alphabet $\Sigma \cup \{\#\}$ such that $L = \{p \in \Sigma^{**} : B_{2,2}(\hat{p}) \subseteq \theta\}$.*

The set θ is usually called a *representation by tiles* for the local language L, and we write $L = L(\theta)$.

Example 1.2 The language of the pictures over $\Sigma = \{0, 1\}$ of square shape with the symbol 0 in one diagonal and the symbol 1 in all the other positions is a local language (see Fig. 2). The representation by tiles is given by:

$$\theta_{D0} = \left\{ \begin{array}{cccccc} \begin{array}{|c|c|}\hline \# & \# \\ \hline \# & 0 \\ \hline \end{array}, & \begin{array}{|c|c|}\hline \# & \# \\ \hline 1 & \# \\ \hline \end{array}, & \begin{array}{|c|c|}\hline \# & 1 \\ \hline \# & \# \\ \hline \end{array}, & \begin{array}{|c|c|}\hline 0 & \# \\ \hline \# & \# \\ \hline \end{array}, & \begin{array}{|c|c|}\hline \# & \# \\ \hline 0 & 1 \\ \hline \end{array}, & \begin{array}{|c|c|}\hline \# & \# \\ \hline 1 & 1 \\ \hline \end{array} \\ \\ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \# & \# \\ \hline \end{array}, & \begin{array}{|c|c|}\hline 1 & 0 \\ \hline \# & \# \\ \hline \end{array}, & \begin{array}{|c|c|}\hline 1 & \# \\ \hline 1 & \# \\ \hline \end{array}, & \begin{array}{|c|c|}\hline 1 & \# \\ \hline 0 & \# \\ \hline \end{array}, & \begin{array}{|c|c|}\hline \# & 0 \\ \hline \# & 1 \\ \hline \end{array}, & \begin{array}{|c|c|}\hline \# & 1 \\ \hline \# & 1 \\ \hline \end{array}, \\ \\ \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array}, & \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 1 & 1 \\ \hline \end{array}, & \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array}, & \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 1 \\ \hline \end{array} & & \end{array} \right\}.$$

The family of local languages will be denoted by LOC. We assume that the empty picture belongs to $L(\theta)$ if and only if θ contains the tile $\begin{array}{|c|c|}\hline \# & \# \\ \hline \# & \# \\ \hline \end{array}$.

Definition 1.3 *A tiling system (TS) is a 4-uple $\mathcal{T} = (\Sigma, \Gamma, \theta, \pi)$, where Σ and Γ are two finite alphabets, θ is a finite set of tiles over the alphabet $\Gamma \cup \{\#\}$, and $\pi : \Gamma \to \Sigma$ is a projection.*

We say that a tiling system $\mathcal{T}$ defines the language $L = \pi(L(\theta))$, where $L(\theta)$ is a local language over Γ, called the *underlying language* for L, and write by convention $L = L(\mathcal{T})$. Moreover, we say that $L \subseteq \Sigma^{**}$ is *recognizable by tiling systems* (or *tiling recognizable*) if there exists a tiling system $\mathcal{T} = (\Sigma, \Gamma, \theta, \pi)$, such that $L = L(\mathcal{T})$. From now on the family of two-dimensional languages recognizable by tiling systems will be denoted by $\mathcal{L}(TS)$.

#	#	#	#	#	#
#	0	1	1	1	#
#	1	0	1	1	#
#	1	1	0	1	#
#	1	1	1	0	#
#	#	#	#	#	#

(a)

#	#	#	#	#	#
#	a	a	a	a	#
#	a	a	a	a	#
#	a	a	a	a	#
#	a	a	a	a	#
#	#	#	#	#	#

(b)

Figure 2: A picture in $L(\theta_{D0})$, (a), and the corresponding picture in $L(\mathcal{T})$, (b).

Example 1.4 Consider the local language in Example 1.2. Let $p : \Sigma \rightarrow \{a\}$ be the projection that maps each element of Σ in a. Let $\Gamma = \{a\}$, then the tiling system $\mathcal{T} = (\Sigma, \Gamma, \theta_{D0}, p)$ recognizes the language $L(\mathcal{T})$ of the pictures over Γ having the form of a square (see Fig. 2). Notice that such a language is not local, while it is tiling recognizable.

1.2 Basics on polyominoes

In the plane $\mathbb{Z} \times \mathbb{Z}$ a *cell* is a unit square, and a *polyomino* is a finite connected union of cells having no cut point. Polyominoes are defined up to translations.

A *column* (*row*) of a polyomino is the intersection between the polyomino and an infinite strip of cells whose centers lie on a vertical (horizontal) line.

In general, problems like enumeration, exhaustive and random generation of polyominoes are difficult to solve and still open. Thus, in order to simplify some of these problems, several subclasses were defined by combining two notions: the geometrical notion of *convexity*, and the notion of *directed growth*, which comes from statistical physics. A polyomino is said to

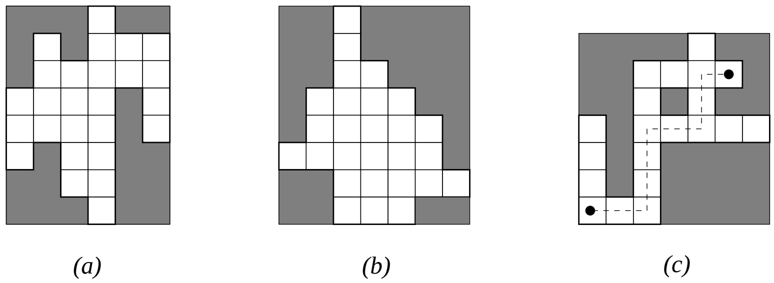


Figure 3: (a) a column-convex polyomino; (b) a convex polyomino; (c) a directed (not convex) polyomino.

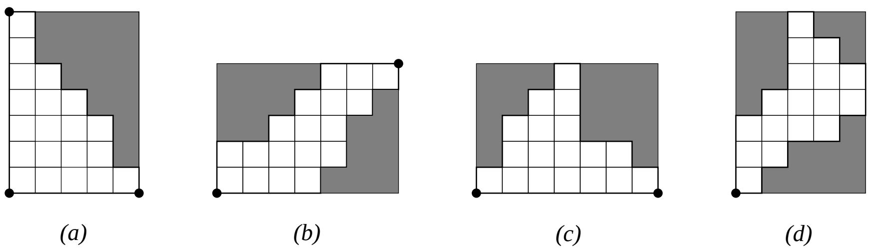


Figure 4: (a) a Ferrers diagram; (c) a parallelogram polyomino; (c) a stack polyomino; (d) a directed-convex polyomino.

be *column-convex* [*row-convex*] when its intersection with any vertical [horizontal] line is convex (Fig. 3 (a)). A polyomino is *convex* if it is both column and row convex (Fig. 3 (b)). A polyomino P is said to be *directed* when every cell of P can be reached from a distinguished cell (usually the leftmost at the lowest ordinate), by a path which is contained in P and only uses north and east unit steps (Fig.3 (c)). Figure 4 (d) depicts a polyomino that is both directed and convex. Moreover we can define three types of directed and convex polyominoes, i.e. the *Ferrers diagrams* (Fig.4 (a)), the *parallelogram polyominoes* (Fig.4 (b)), and the *stack polyominoes* (Fig.4 (c)). As Figure 4 shows, each of these three subsets can be characterized, in the set of convex polyominoes, by the fact that two or three vertices of the minimal bounding rectangle of the polyomino must also belong to the polyomino itself.

In this paper we also deal with a special class of convex polyominoes, the *L-convex polyominoes*, introduced in [CR03] as a first level in a classification of convex polyominoes.

To define this class we must start giving some basic concepts. In a polyomino an *internal path* is a self-avoiding sequence of unitary steps of four types: north $N = (0,1)$, south $S = (0,-1)$, east $E = (1,0)$, and west $W = (-1,0)$. We say that a path is *monotone* if it is made with steps of only two types. The authors of [CR03] observed that convex polyominoes have the property that every pair of cells is connected by a monotone path entirely contained in the polyomino. In this way each convex polyomino is characterized by a parameter k that represents the minimal number of changes of direction in these paths. More precisely, a convex polyomino is called k-*convex* if, for every pair of its cells, there is at least a monotone path with at most k changes of direction that connects them. When the value of k is 1 we have the so called L-convex polyominoes, where this terminology is motivated by the L-shape of the path that connects any two cells (see Figure 5).

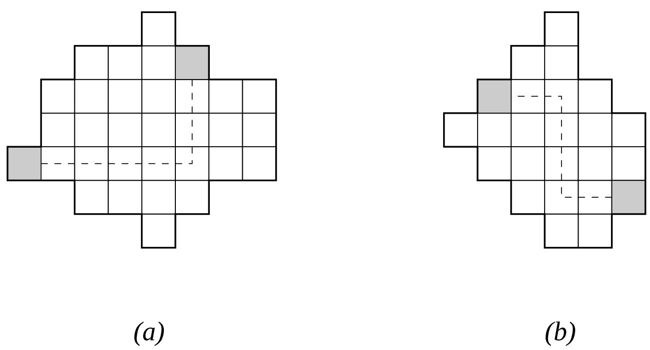


Figure 5: (a) an L-convex polyomino, and a monotone path with a single change of direction joining two of its cells; (b) a convex but not L-convex polyomino: the two highlighted cells cannot be connected by a path with only one change of direction.

This class of polyominoes has been successively considered by several points of view: in [CR05] it is shown that L-convex polyominoes are a well-ordering according to the sub-picture order, in [CFRR] the authors have investigated some tomographical aspects of this family, and have discovered that L-convex polyominoes are uniquely determined by their horizontal and vertical projections. Finally, in [CFRR2] it is proved that the number f_n of L-convex polyominoes having semi-perimeter equal to $n+2$ satisfies the recurrence relation:

$$f_n = 4f_{n-1} - 2f_{n-2}, \qquad n \geq 3, \tag{1}$$

with $f_0 = 1$, $f_1 = 2$, $f_2 = 7$. Successively [CFMRR] the authors have studied the problem of enumerating L-convex polyominoes by the area, and provided a coding of L-convex polyominoes in terms of words of a regular language.

1.3 Contents of the paper

In Section 2 we prove that many classes of convex polyominoes, among which the ones most commonly treated in the literature, can be encoded as words of tiling recognizable two-dimensional languages. This statement is achieved by providing a set of tiles for each of these languages, and proving that convexity constrains can be formulated by means of local properties of the boundary of the polyomino.

In Section 3 we consider L-convex polyominoes, which, differently from the other classes of convex polyominoes treated in Section 2, are not defined by a "local" property on the boundary. However, we prove that also L-convex polyominoes can be recognized by a tiling system.

Finally, in Section 4, we deal with some properly enumerative problems: we define the generating function of a two-dimensional language, and we investigate the nature and the analytical properties of the generating functions of tiling systems.

We recall that for a string language $\mathcal{L}$, the generating function $f_{\mathcal{L}}(x)$ is the formal power series $f_{\mathcal{L}}(x) = \sum_{n \geq 0} f_n x^n$, such that for all $n \in \mathbb{N}$,

$$f_n = \|\{w \in \mathcal{L} : |w| = n\}\|,$$

where $|w|$ denotes the length of w. By classical result of Chomsky and Schützenberger [CS] we have that the generating function of a regular language is rational.

It is natural to investigate whether such a characterization still holds when we pass to the two-dimensional case, in particular for the class of tiling systems, that constitutes the two-dimensional counterpart of regular languages. We show that, quite surprisingly, the generating functions of tiling systems are not necessarily rational, but can also be algebraic, D-finite, and non D-finite. This fact certifies that tiling systems are capable of representing a large amount of combinatorial structures classes which cannot be handled using string languages.

2 Polyominoes and tiling systems

In this section we will prove that some classes of polyominoes are represented by tiling recognizable languages. Let us consider the following two-dimensional languages on the alphabet $\{0, 1\}$: $\mathcal{C}$ (resp. $\mathcal{F}$, $\mathcal{S}$, $\mathcal{P}$, $\mathcal{D}$, $\mathcal{V}$) is the class of pictures that represent convex polyominoes (resp. Ferrers diagrams, stack polyominoes, parallelogram polyominoes, directed-convex polyominoes, column-convex polyominoes). We will first prove that $\mathcal{C}$ is a tiling recognizable language, and, as a consequence, that $\mathcal{F}$ is a local language and that $\mathcal{S}$, $\mathcal{P}$, $\mathcal{D}$, and $\mathcal{V}$ are tiling recognizable languages.

Let P be a convex polyomino, $R(P)$ be its minimal bounding rectangle; we start by observing that four disjoint (possibly empty) sets of unit cells in $R(P) \backslash P$ are easily individuated, each of them located at one of the four vertices of $R(P)$. Let us call these sets A, B, C, and D (see Fig. 6 (a)). An easy check reveals the following property.

Proposition 2.1 *P is convex if and only if for each cell (i, j) of $R(P)$ it holds*

- *if $(i, j) \in A$, $(i-1, j)$ and $(i, j-1)$ belong to A or lie on the boundary of P;*
- *if $(i, j) \in B$ then $(i-1, j)$, $(i, j+1)$ belong to B or lie on the boundary of P;*
- *if $(i, j) \in C$ then $(i+1, j)$, $(i, j-1)$ belong to C or lie on the boundary of P;*
- *if $(i, j) \in D$ then $(i+1, j)$, $(i, j+1)$ belong to C or lie on the boundary of P.*

The reader can easily check the previous proposition by observing Figure 6.

To each convex polyomino we associate a picture obtained by representing with a 1 every cell belonging to the polyomino, and with the symbol a (resp. b, c, d) every cell in A (resp. B, C, D), as depicted in Fig. 6 (b). Let $\mathcal{L}_C$ be the language of these rectangles over the alphabet $\{1, a, b, c, d\}$.

Let us now consider the following sets of tiles:

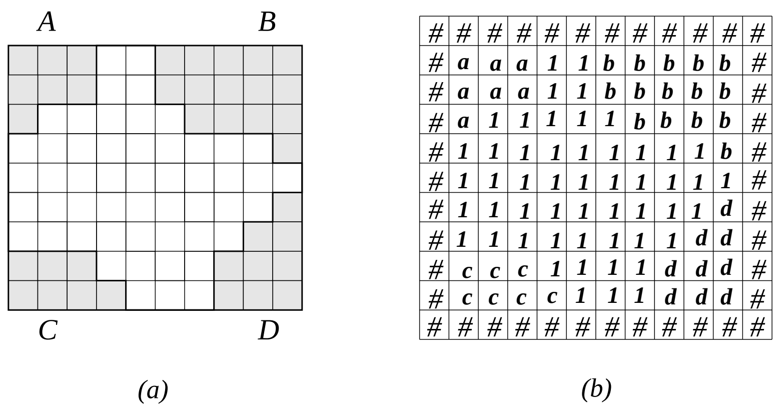


Figure 6: (a) a convex polyomino P individuates four disjoint sets of cells in $R(P)\setminus P$; (b) the representation of P as a word of $\mathcal{L}_C$.

$$\theta_R = \left\{ \begin{array}{ccccc} \begin{array}{|c|c|}\hline \# & \# \\ \hline \# & \# \\ \hline \end{array}, & \begin{array}{|c|c|}\hline \# & \# \\ \hline \# & 1 \\ \hline \end{array}, & \begin{array}{|c|c|}\hline \# & \# \\ \hline 1 & \# \\ \hline \end{array}, & \begin{array}{|c|c|}\hline \# & 1 \\ \hline \# & \# \\ \hline \end{array}, & \begin{array}{|c|c|}\hline 1 & \# \\ \hline \# & \# \\ \hline \end{array}, \\ \\ \begin{array}{|c|c|}\hline \# & \# \\ \hline 1 & 1 \\ \hline \end{array}, & \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \# & \# \\ \hline \end{array}, & \begin{array}{|c|c|}\hline \# & 1 \\ \hline \# & 1 \\ \hline \end{array}, & \begin{array}{|c|c|}\hline 1 & \# \\ \hline 1 & \# \\ \hline \end{array}, & \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 1 \\ \hline \end{array} \end{array} \right\},$$

This is the set of tiles necessary to recognize *rectangular polyominoes*.

$$\theta_A = \left\{ \begin{array}{ccccc} \begin{array}{|c|c|}\hline a & a \\ \hline a & a \\ \hline \end{array}, & \begin{array}{|c|c|}\hline \# & \# \\ \hline \# & a \\ \hline \end{array}, & \begin{array}{|c|c|}\hline \# & \# \\ \hline a & a \\ \hline \end{array}, & \begin{array}{|c|c|}\hline \# & a \\ \hline \# & a \\ \hline \end{array}, & \begin{array}{|c|c|}\hline \# & \# \\ \hline a & 1 \\ \hline \end{array}, \\ \\ \begin{array}{|c|c|}\hline \# & a \\ \hline \# & 1 \\ \hline \end{array}, & \begin{array}{|c|c|}\hline a & a \\ \hline a & 1 \\ \hline \end{array}, & \begin{array}{|c|c|}\hline a & 1 \\ \hline a & 1 \\ \hline \end{array}, & \begin{array}{|c|c|}\hline a & a \\ \hline 1 & 1 \\ \hline \end{array}, & \begin{array}{|c|c|}\hline a & 1 \\ \hline 1 & 1 \\ \hline \end{array} \end{array} \right\},$$

$$\theta_B = \left\{ \begin{array}{ccccc} \begin{array}{|c|c|}\hline b & b \\ \hline b & b \\ \hline \end{array}, & \begin{array}{|c|c|}\hline \# & \# \\ \hline b & \# \\ \hline \end{array}, & \begin{array}{|c|c|}\hline \# & \# \\ \hline b & b \\ \hline \end{array}, & \begin{array}{|c|c|}\hline b & \# \\ \hline b & \# \\ \hline \end{array}, & \begin{array}{|c|c|}\hline \# & \# \\ \hline 1 & b \\ \hline \end{array}, \\ \\ \begin{array}{|c|c|}\hline b & \# \\ \hline 1 & \# \\ \hline \end{array}, & \begin{array}{|c|c|}\hline b & b \\ \hline 1 & b \\ \hline \end{array}, & \begin{array}{|c|c|}\hline 1 & b \\ \hline 1 & b \\ \hline \end{array}, & \begin{array}{|c|c|}\hline b & b \\ \hline 1 & 1 \\ \hline \end{array}, & \begin{array}{|c|c|}\hline 1 & b \\ \hline 1 & 1 \\ \hline \end{array} \end{array} \right\},$$

$$\theta_C = \left\{ \begin{array}{ccccc} \begin{array}{|c|c|}\hline c & c \\ \hline c & c \\ \hline \end{array}, & \begin{array}{|c|c|}\hline \# & c \\ \hline \# & \# \\ \hline \end{array}, & \begin{array}{|c|c|}\hline c & c \\ \hline \# & \# \\ \hline \end{array}, & \begin{array}{|c|c|}\hline \# & c \\ \hline \# & c \\ \hline \end{array}, & \begin{array}{|c|c|}\hline c & 1 \\ \hline \# & \# \\ \hline \end{array}, \\ \\ \begin{array}{|c|c|}\hline \# & 1 \\ \hline \# & c \\ \hline \end{array}, & \begin{array}{|c|c|}\hline c & 1 \\ \hline c & c \\ \hline \end{array}, & \begin{array}{|c|c|}\hline c & 1 \\ \hline c & 1 \\ \hline \end{array}, & \begin{array}{|c|c|}\hline 1 & 1 \\ \hline c & c \\ \hline \end{array}, & \begin{array}{|c|c|}\hline 1 & 1 \\ \hline c & 1 \\ \hline \end{array} \end{array} \right\},$$

$$\theta_D = \left\{ \begin{array}{ccccc} \begin{array}{|c|c|}\hline d & d \\ \hline d & d \\ \hline\end{array}, & \begin{array}{|c|c|}\hline d & \# \\ \hline \# & \# \\ \hline\end{array}, & \begin{array}{|c|c|}\hline d & \# \\ \hline d & \# \\ \hline\end{array}, & \begin{array}{|c|c|}\hline d & d \\ \hline \# & \# \\ \hline\end{array}, & \begin{array}{|c|c|}\hline 1 & d \\ \hline \# & \# \\ \hline\end{array}, \\ \\ \begin{array}{|c|c|}\hline 1 & \# \\ \hline d & \# \\ \hline\end{array}, & \begin{array}{|c|c|}\hline 1 & d \\ \hline d & d \\ \hline\end{array}, & \begin{array}{|c|c|}\hline 1 & d \\ \hline 1 & d \\ \hline\end{array}, & \begin{array}{|c|c|}\hline 1 & 1 \\ \hline d & d \\ \hline\end{array}, & \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & d \\ \hline\end{array} \end{array} \right\}.$$

It is easy to prove that the sets θ_A, θ_B, θ_C, and θ_D realize the conditions of Proposition 2.1 with respect to the cells of A, B, C, and D, respectively, and so, together with θ_R which characterizes the internal part of P, they allow the following:

Theorem 2.2 *$\mathcal{L}_C$ is a local language over the alphabet $\Sigma_C = \{1, a, b, c, d\}$, and $\mathcal{L}_C = L(\theta_R \cup \theta_A \cup \theta_B \cup \theta_C \cup \theta_D)$.*

The proof is omitted for brevity.

Let $\mathcal{L}_F = L(\theta_R \cup \theta_B)$; since easily we have that $\mathcal{L}_F = \mathcal{F}$, $\mathcal{F}$ is a local language (and, consequently, tiling system recognizable). Furthermore, let us consider the following local languages:

- $\mathcal{L}_S = L(\theta_R \cup \theta_A \cup \theta_B)$, over $\Sigma_S = \{1, a, b\}$;
- $\mathcal{L}_P = L(\theta_R \cup \theta_A \cup \theta_D)$, over $\Sigma_P = \{1, a, d\}$;
- $\mathcal{L}_D = L(\theta_R \cup \theta_A \cup \theta_B \cup \theta_D)$, over $\Sigma_D = \{1, a, b, d\}$,

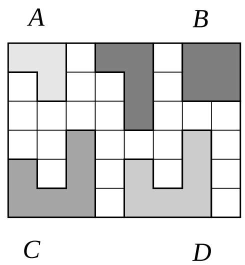
and the projections:

- $\pi_S : \Sigma_S \to \{0, 1\}$, such that $\pi_S(a) = \pi_S(b) = 0$, $\pi_S(1) = 1$;
- $\pi_P : \Sigma_P \to \{0, 1\}$, such that $\pi_P(a) = \pi_P(d) = 0$, $\pi_P(1) = 1$;
- $\pi_D : \Sigma_D \to \{0, 1\}$, such that $\pi_D(a) = \pi_D(b) = \pi_D(d) = 0$, $\pi_D(1) = 1$,

we finally have that:

- $\pi_S(\mathcal{L}_S) = \mathcal{S}$, thus $\mathcal{S}$ is tiling recognizable;
- $\pi_P(\mathcal{L}_P) = \mathcal{P}$, thus $\mathcal{P}$ is tiling recognizable;
- $\pi_D(\mathcal{L}_D) = \mathcal{D}$, thus $\mathcal{D}$ is tiling recognizable.

The proof that $\mathcal{V}$ is tiling recognizable resembles the previous proofs. Let P be a column-convex polyomino, and $R(P)$ its minimal bounding rectangle; two disjoint (possibly empty) sets of unit cells in $R(P)\backslash P$ can now be easily individuated: one comprehends the cells above P, and the other comprehends the cells below P. Each of these two zones is further divided into



(a)

#	#	#	#	#	#	#	#	#	#
#	***a***	***a***	***1***	***b***	***b***	***1***	***b***	***b***	#
#	***1***	***a***	***1***	***1***	***b***	***1***	***b***	***b***	#
#	***1***	***1***	***1***	***1***	***b***	***1***	***1***	***1***	#
#	***1***	***1***	***c***	***1***	***1***	***1***	***d***	***1***	#
#	***c***	***1***	***c***	***1***	***d***	***1***	***d***	***1***	#
#	***c***	***c***	***c***	***1***	***d***	***d***	***d***	***1***	#
#	#	#	#	#	#	#	#	#	#

(b)

Figure 7: (a) a column-convex polyomino P individuates four disjoint sets of cells in $R(P)\backslash P$; (b) the representation of P as a word of $\mathcal{L}_V$.

two sets: the leftmost set of the upper [resp. lower] zone is still indicated by A [resp. C], and its remaining part by B [resp. D], as shown in Fig. 7 (a). Let us now consider the language $\mathcal{L}_V$ of rectangles over the alphabet $\{1, a, b, c, d\}$ obtained representing each convex polyomino as follows: a cell belonging to the polyomino is coded by 1, each cell in A (resp. B, C, D) is coded by a (resp. b, c, d), as depicted in Fig. 7 (b).

Proposition 2.3 *$\mathcal{L}_V$ is a local language, i.e.* $\mathcal{L}_V = L\left(\theta_R \cup \theta'_A \cup \theta'_B \cup \theta'_C \cup \theta'_D\right)$, *where*

$$\theta'_A = \theta_A \cup \left\{ \begin{array}{|c|c|}\hline a & a \\ \hline 1 & a \\ \hline \end{array}\ , \begin{array}{|c|c|}\hline 1 & a \\ \hline 1 & a \\ \hline \end{array}\ , \begin{array}{|c|c|}\hline 1 & a \\ \hline 1 & 1 \\ \hline \end{array} \right\},$$

$$\theta'_B = \theta_B \cup \left\{ \begin{array}{|c|c|}\hline \# & \# \\ \hline b & 1 \\ \hline \end{array}\ , \begin{array}{|c|c|}\hline b & b \\ \hline b & 1 \\ \hline \end{array}\ , \begin{array}{|c|c|}\hline b & 1 \\ \hline b & 1 \\ \hline \end{array}\ , \begin{array}{|c|c|}\hline b & 1 \\ \hline 1 & 1 \\ \hline \end{array} \right\},$$

$$\theta'_C = \theta_C \cup \left\{ \begin{array}{|c|c|}\hline 1 & c \\ \hline c & c \\ \hline \end{array}\ , \begin{array}{|c|c|}\hline 1 & c \\ \hline 1 & c \\ \hline \end{array}\ , \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & c \\ \hline \end{array} \right\},$$

$$\theta'_D = \theta_D \cup \left\{ \begin{array}{|c|c|}\hline d & 1 \\ \hline \# & \# \\ \hline \end{array}\ , \begin{array}{|c|c|}\hline d & 1 \\ \hline d & d \\ \hline \end{array}\ , \begin{array}{|c|c|}\hline d & 1 \\ \hline d & 1 \\ \hline \end{array}\ , \begin{array}{|c|c|}\hline 1 & d \\ \hline 1 & 1 \\ \hline \end{array} \right\}.$$

Finally, we have that $\pi_V(\mathcal{L}_V) = \mathcal{V}$, where π_V is a projection from $\{1, a, b, c, d\}$ to $\{0, 1\}$ defined as: $\pi_V(a) = \pi_V(b) = \pi_V(b) = \pi_V(b) = 0$, $\pi_V(1) = 1$. Thus $\mathcal{V}$ is tiling recognizable.

3 Tiling recognizability of L-convex polyominoes

Let us focus our attention on the two dimensional language $\mathcal{L}_{Conv}$ on the alphabet $\{0, 1\}$, which represents the class of L-convex polyominoes. After recalling their characterization in terms of maximal rectangles given in [CR03], we proceed in studying the tiling recognizability of $\mathcal{L}_{Conv}$.

By abuse of notation, for any two polyominoes P and P' we will write $P \subseteq P'$ to mean that P is geometrically included in P'. An $x \times y$ *rectangle*, with $x, y \in \mathbb{N} \setminus \{0\}$, is a rectangular polyomino with x columns and y rows. We say an $x \times y$ rectangle to be *maximal* in P if it is not properly contained in any other rectangle entirely contained in P.

Two $x \times y$ and $x' \times y'$ rectangles have *crossing intersection* if their intersection is a $min(x, x') \times min(y, y')$ rectangle, i.e. a rectangle whose basis is the smallest of the two bases and whose height is the smallest of the two heights (see Fig. 8).

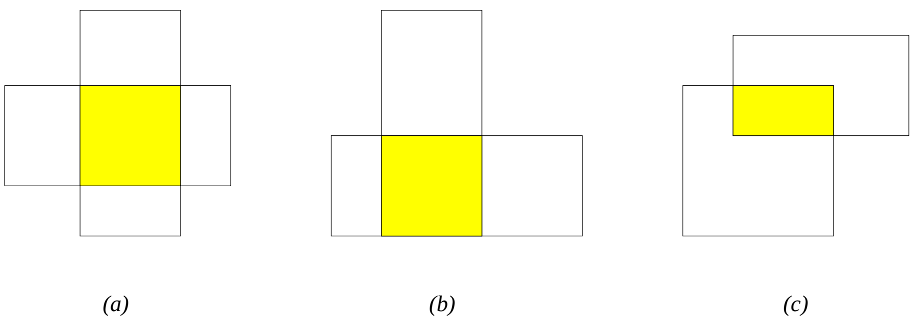


Figure 8: The two rectangles in (a) and (b) have crossing intersection, while the two in (c) does not.

Now we remind a theorem which furnishes a useful characterization of L-convex polyominoes in terms of maximal rectangles [CR03], and which relies on the following immediate lemma:

Lemma 3.1 *An L-convex polyominoes does not contain two different maximal rectangles of the same dimensions.*

Theorem 3.2 *A convex polyomino P is L-convex if and only if, any two maximal rectangles of P have crossing intersection (see Fig. 9).*

Now we pass to construct the tiling system for the class of L-convex polyominoes; the idea is the following: we label each cell of the polyomino

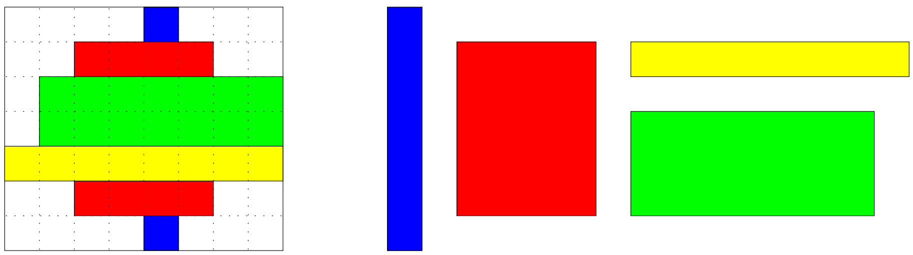


Figure 9: The L-convex polyomino in Fig. 5 (a), on the left, can be obtained as the union of four maximal rectangles on the right, any two of them having crossing intersection.

we are going to recognize with a couple (x, y) of natural numbers, and each cell outside the polyomino with the letters a, b, c, and d following the same criterion adopted for convex polyominoes.

The tiling system must assure that two different cells of a polyomino belong to the intersection of the same set of maximal rectangles, if and only if their labels have the same value.

More precisely, we divide the cells of an L-convex polyominoes into connected sets (zones) according to the number of maximal rectangles they belong to, and we proceed by labelling with $(0, 0)$ each cell belonging to the central zone of the polyomino, i.e. where all the maximal rectangles intersect. Then, moving to the left (resp. to the right) of that zone, we increase (resp. decrease) the first element of the couple according to the difference of the number of maximal rectangles of the two zones, as it will be explained later. In a similar way, we act on the second element of the couple when moving up or down the central zone. Applying recursively that procedure, we assign to each cell of the polyomino a couple of integers, as desired. Before going deep into detail, the reader can check his intuition of the process looking at Fig. 10.

So, let $\mathcal{L}^k_{Conv}$ be the class of (pictures representing) L-convex polyominoes having at most k maximal rectangles. Clearly it holds $\bigcup_{k\geq 1} \mathcal{L}^k_{Conv} = \mathcal{L}_{Conv}$. Our aim is to prove that, for each $k \geq 2$, $\mathcal{L}^k_{Conv}$ is tiling recognizable (when $k = 1$ the assumption trivially holds).

Let us enrich the alphabet used for convex polyominoes and define

$$\Gamma_k = \{(x, y) \mid x, y \in \mathbb{Z},\ |x|, |y| < k\} \cup \{a, b, c, d\}.$$

As one can expect, the couple of integers inserted in Γ_k turns out to be sufficient to label each cell of an L-convex polyomino having k maximal rectangles at most, while the symbols a, b, c, and d will be used to label each cell in the zones A, B, C, and D of the minimal bounding rectangle outside the polyomino, respectively.

For each $0 \leq x_1, y_1, i, j \leq k-1$, and $-k+1 \leq x_2, y_2 \leq 0$, let us consider the following sets of tiles:

$$\theta_A^k = \left\{ \begin{array}{l} \begin{array}{|c|c|}\hline \# & \# \\ \hline \# & a \\ \hline\end{array}, \begin{array}{|c|c|}\hline \# & \# \\ \hline a & a \\ \hline\end{array}, \begin{array}{|c|c|}\hline \# & a \\ \hline \# & a \\ \hline\end{array}, \begin{array}{|c|c|}\hline \# & \# \\ \hline a & (0, y_1+1) \\ \hline\end{array}, \begin{array}{|c|c|}\hline \# & a \\ \hline \# & (x_2-1, 0) \\ \hline\end{array}, \\ \\ \begin{array}{|c|c|}\hline (x_2-i, y_1+j) & (x_2, y_1+j) \\ \hline (x_2-i, y_1) & (x_2, y_1) \\ \hline\end{array} \end{array} \right\},$$

$$\theta_B^k = \left\{ \begin{array}{l} \begin{array}{|c|c|}\hline \# & \# \\ \hline b & \# \\ \hline\end{array}, \begin{array}{|c|c|}\hline \# & \# \\ \hline b & b \\ \hline\end{array}, \begin{array}{|c|c|}\hline b & \# \\ \hline b & \# \\ \hline\end{array}, \begin{array}{|c|c|}\hline \# & \# \\ \hline (0, y_1+1) & b \\ \hline\end{array}, \begin{array}{|c|c|}\hline b & \# \\ \hline (x_1+1, 0) & \# \\ \hline\end{array}, \\ \\ \begin{array}{|c|c|}\hline (x_1, y_1+j) & (x_1+i, y_1+j) \\ \hline (x_1, y_1) & (x_1+i, y_1) \\ \hline\end{array} \end{array} \right\},$$

$$\theta_C^k = \left\{ \begin{array}{l} \begin{array}{|c|c|}\hline \# & c \\ \hline \# & \# \\ \hline\end{array}, \begin{array}{|c|c|}\hline c & c \\ \hline \# & \# \\ \hline\end{array}, \begin{array}{|c|c|}\hline \# & c \\ \hline \# & c \\ \hline\end{array}, \begin{array}{|c|c|}\hline c & (0, y_2-1) \\ \hline \# & \# \\ \hline\end{array}, \begin{array}{|c|c|}\hline \# & (x_2-1, 0) \\ \hline \# & c \\ \hline\end{array}, \\ \\ \begin{array}{|c|c|}\hline (x_2-i, y_2) & (x_2, y_2) \\ \hline (x_2-i, y_2-j) & (x_2, y_2-j) \\ \hline\end{array} \end{array} \right\},$$

$$\theta_D^k = \left\{ \begin{array}{l} \begin{array}{|c|c|}\hline d & \# \\ \hline \# & \# \\ \hline\end{array}, \begin{array}{|c|c|}\hline d & d \\ \hline \# & \# \\ \hline\end{array}, \begin{array}{|c|c|}\hline d & \# \\ \hline d & \# \\ \hline\end{array}, \begin{array}{|c|c|}\hline (0, y_2-1) & d \\ \hline \# & \# \\ \hline\end{array}, \begin{array}{|c|c|}\hline (x_1+1, 0) & \# \\ \hline d & \# \\ \hline\end{array}, \\ \\ \begin{array}{|c|c|}\hline (x_1, y_2) & (x_1+i, y_2) \\ \hline (x_1, y_2-j) & (x_1+i, y_2-j) \\ \hline\end{array} \end{array} \right\},$$

$$
\theta_R^k = \left\{ \begin{array}{l}
\begin{array}{|c|c|}\hline \# & \# \\ \hline \# & (x_2,y_1) \\ \hline \end{array}, \quad
\begin{array}{|c|c|}\hline \# & \# \\ \hline (x_2-i,y_1) & (x_2,y_1) \\ \hline \end{array}, \quad
\begin{array}{|c|c|}\hline \# & \# \\ \hline (x_1,y_1) & (x_1+i,y_1) \\ \hline \end{array}, \quad
\begin{array}{|c|c|}\hline \# & \# \\ \hline (x_1,y_1) & \# \\ \hline \end{array}, \\[2ex]
\begin{array}{|c|c|}\hline \# & (x_2,y_2) \\ \hline \# & \# \\ \hline \end{array}, \quad
\begin{array}{|c|c|}\hline (x_2-i,y_2) & (x_2,y_2) \\ \hline \# & \# \\ \hline \end{array}, \quad
\begin{array}{|c|c|}\hline (x_1,y_2) & (x_1+i,y_2) \\ \hline \# & \# \\ \hline \end{array}, \quad
\begin{array}{|c|c|}\hline (x_1,y_2) & \# \\ \hline \# & \# \\ \hline \end{array}, \\[2ex]
\begin{array}{|c|c|}\hline \# & (x_2,y_1+j) \\ \hline \# & (x_2,y_1) \\ \hline \end{array}, \quad
\begin{array}{|c|c|}\hline \# & (x_2,y_2) \\ \hline \# & (x_2,y_2-j) \\ \hline \end{array}, \quad
\begin{array}{|c|c|}\hline (x_1,y_1+j) & \# \\ \hline (x_1,y_1) & \# \\ \hline \end{array}, \quad
\begin{array}{|c|c|}\hline (x_1,y_2) & \# \\ \hline (x_1,y_2-j) & \# \\ \hline \end{array}, \\[2ex]
\begin{array}{|c|c|}\hline (x_2-i,y_1+j) & (x_2,y_1+j) \\ \hline (x_2-i,y_1) & (x_2,y_1) \\ \hline \end{array}, \quad
\begin{array}{|c|c|}\hline (x_1,y_1+j) & (x_1+i,y_1+j) \\ \hline (x_1,y_1) & (x_1+i,y_1) \\ \hline \end{array}, \\[2ex]
\begin{array}{|c|c|}\hline (x_2-i,y_2) & (x_2,y_2) \\ \hline (x_2-i,y_2-j) & (x_2,y_2-j) \\ \hline \end{array}, \quad
\begin{array}{|c|c|}\hline (x_1,y_2) & (x_1+i,y_2) \\ \hline (x_1,y_2-j) & (x_1+i,y_2-j) \\ \hline \end{array}
\end{array} \right\},
$$

provided with the following constraints:

- if the element (x,y) belongs to a tile in θ_A^k (resp. θ_B^k, θ_C^k, and θ_D^k), with $|x|+|y| \geq k$, then (x,y) is replaced by a (resp. b, c, and d);
- each tile in θ_A^k (resp. θ_B^k, θ_C^k, and θ_D^k) must contain at least one element a (resp. b, c, and d);
- the elements (x,y) of the tiles in θ_R^k must satisfy $|x|+|y| < k$.

Notice that these sets are clearly redundant, in the sense that the intersection of two of them is not necessarily empty, but for the sake of readability we prefer to maintain this description.

Lemma 3.3 *Let $\varphi : \Gamma_k \to \Gamma_C$ be the projection such that $\varphi((x,y)) = 1$, $\varphi(a) = a$, $\varphi(b) = b$, $\varphi(c) = c$, and $\varphi(d) = d$. It holds that*

$$
\varphi(\theta_A^k) = \theta_A, \;\; \varphi(\theta_B^k) = \theta_B, \;\; \varphi(\theta_C^k) = \theta_C, \;\; \varphi(\theta_D^k) = \theta_D, \quad \textit{and} \quad \varphi(\theta_R^k) = \theta_R,
$$

where the sets θ_A, θ_B, θ_C, θ_D, and θ_R have been considered in the previous section.

By definition of tiling system, it follows that each element of

$$
\mathcal{L}_{Conv}^k = L\left(\theta_A^k \cup \theta_B^k \cup \theta_C^k \cup \theta_D^k \cup \theta_R^k\right)
$$

can be mapped into a convex polyomino, i.e. $\mathcal{L}_{Conv}^k \subset \mathcal{L}_C$, for each $k \geq 1$.

Lemma 3.4 *Let $1 \leq h \leq k$, and let $\varphi : \Gamma_k \to \Gamma_{k-h}$ be the projection such that $\varphi(a) = a$, $\varphi(b) = b$, $\varphi(c) = c$, $\varphi(d) = d$, and*

$$\varphi\left((x,y) \right) = \begin{cases} (x,y), & \text{if } |x| + |y| < k-h; \\ a, & \text{if } -x+y \geq k-h; \\ b, & \text{if } x+y \geq k-h; \\ c, & \text{if } -x-y \geq k-h; \\ d, & \text{if } x-y \geq k-h. \end{cases}$$

It holds that

$$\varphi(\theta_A^k) = \theta_A^{k-h}, \varphi(\theta_B^k) = \theta_B^{k-h}, \varphi(\theta_C^k) = \theta_C^{k-h}, \varphi(\theta_D^k) = \theta_D^{k-h}, \text{ and } \varphi(\theta_R^k) = \theta_R^{k-h}.$$

As an immediate consequence of Lemma 3.4, we have $\mathcal{L}_{Conv}^{k-h} \subset \mathcal{L}_{Conv}^{k}$. In practice the projection φ works on each set θ_A^k (resp. θ_B^k, θ_C^k, θ_D^k) leaving unaltered the tiles which are also tiles of θ_A^{k-h} (resp. θ_B^{k-h}, θ_C^{k-h}, θ_D^{k-h}), and setting all the others equal to a (resp. b, c, d).

The following example will clarify the construction of the sets of tiles:

Example 3.5 *Let us set $k = 2$. We explicitly describe the sets θ_A^2, we partially list θ_R^2 (allowing redundances), and we leave θ_B^2, θ_C^2, and θ_D^2 as a simple exercise:*

$$\theta_A^2 = \left\{ \begin{array}{c} \begin{array}{|c|c|}\hline \# & \# \\ \hline \# & a \\ \hline \end{array}, \; \begin{array}{|c|c|}\hline \# & \# \\ \hline a & a \\ \hline \end{array}, \; \begin{array}{|c|c|}\hline \# & a \\ \hline \# & a \\ \hline \end{array}, \; \begin{array}{|c|c|}\hline \# & \# \\ \hline a & (0,1) \\ \hline \end{array}, \; \begin{array}{|c|c|}\hline \# & \# \\ \hline a & (-1,0) \\ \hline \end{array}, \\ \\ \begin{array}{|c|c|}\hline a & (0,1) \\ \hline (-1,0) & (0,0) \\ \hline \end{array}, \; \begin{array}{|c|c|}\hline a & (0,1) \\ \hline a & (0,1) \\ \hline \end{array}, \; \begin{array}{|c|c|}\hline a & a \\ \hline (-1,0) & (-1,0) \\ \hline \end{array}, \; \begin{array}{|c|c|}\hline a & a \\ \hline a & a \\ \hline \end{array} \end{array} \right\},$$

$$
\theta_R^2 = \left\{
\begin{array}{l}
\begin{array}{|c|c|}\hline \# & \# \\ \hline \# & (-1,0) \\ \hline \end{array}\,, \;
\begin{array}{|c|c|}\hline \# & \# \\ \hline \# & (0,0) \\ \hline \end{array}\,, \;
\begin{array}{|c|c|}\hline \# & \# \\ \hline (-1,0) & (-1,0) \\ \hline \end{array}\,, \;
\begin{array}{|c|c|}\hline \# & \# \\ \hline (-1,0) & (0,0) \\ \hline \end{array}\,, \\[1em]
\begin{array}{|c|c|}\hline \# & \# \\ \hline (0,0) & (0,0) \\ \hline \end{array}\,, \;
\begin{array}{|c|c|}\hline \# & \# \\ \hline (0,0) & (1,0) \\ \hline \end{array}\,, \;
\begin{array}{|c|c|}\hline \# & \# \\ \hline (1,0) & (1,0) \\ \hline \end{array}\,, \; \ldots, \\[1em]
\begin{array}{|c|c|}\hline (0,0) & (0,0) \\ \hline (0,0) & (0,0) \\ \hline \end{array}\,, \;
\begin{array}{|c|c|}\hline (0,1) & (0,1) \\ \hline (0,0) & (0,0) \\ \hline \end{array}\,, \;
\begin{array}{|c|c|}\hline (-1,0) & (0,0) \\ \hline (-1,0) & (0,0) \\ \hline \end{array}\,, \;
\begin{array}{|c|c|}\hline (0,1) & (0,1) \\ \hline (0,1) & (0,1) \\ \hline \end{array}\,, \\[1em]
\begin{array}{|c|c|}\hline (-1,0) & (-1,0) \\ \hline (-1,0) & (-1,0) \\ \hline \end{array}\,, \;
\begin{array}{|c|c|}\hline (0,0) & (0,0) \\ \hline (0,0) & (0,0) \\ \hline \end{array}\,, \;
\begin{array}{|c|c|}\hline (1,0) & (1,0) \\ \hline (0,0) & (0,0) \\ \hline \end{array}\,, \;
\begin{array}{|c|c|}\hline (0,0) & (0,1) \\ \hline (0,0) & (0,1) \\ \hline \end{array}\,, \\[1em]
\begin{array}{|c|c|}\hline (0,1) & (0,1) \\ \hline (0,1) & (0,1) \\ \hline \end{array}\,, \;
\begin{array}{|c|c|}\hline (1,0) & (1,0) \\ \hline (1,0) & (1,0) \\ \hline \end{array}\,, \;
\begin{array}{|c|c|}\hline (0,0) & (0,0) \\ \hline (0,0) & (0,0) \\ \hline \end{array}\,, \; \ldots
\end{array}
\right\},
$$

We have finally collected all the tools for proving the main result of this section:

Theorem 3.6 *For each $k \geq 2$, the language $\mathcal{L}^k_{Conv}$ is tiling recognizable.*

Proof.

Let us define the tiling system $\mathcal{T} = \big(\{0,1\}, \Gamma_k, \theta^k, \varphi\big)$, with

$$\theta^k = \theta^k_A \cup \theta^k_B \cup \theta^k_C \cup \theta^k_D \cup \theta^k_R,$$

and $\varphi : \Gamma_k \to \{0,1\}$, such that $\varphi(a) = \varphi(b) = \varphi(c) = \varphi(d) = 0$, and, for each couple $(x,y) \in \Gamma_k$, $\varphi((x,y)) = 1$.

The thesis is achieved after proving:

a) if P is an L-convex containing at most k maximal rectangles, then there exists an element in $L(\theta^k)$ which represents it;

b) each element of $L(\theta^k)$ represents an L-convex polyomino;

c) each element of $L(\theta^k)$ represents an L-convex polyomino having k maximal rectangles at most.

a) By Lemma 3.4, we assume the polyomino P to contain k maximal rectangles, and we describe how to represent it by means of a picture on Γ_k. Let $r_0 < r_1 < \cdots < r_{k-1}$ be the maximal rectangles of P ordered according to the length of the basis (numbers of cells for each row in a rectangle).

To each cell of $r_h \cap r_{h+1} \cap \cdots \cap r_{h'}$, we associate the label $(x, y) \in \Gamma_k$, such that $|x| = h$ and $|y| = k - 1 - h'$. The signs of x and y are determined by the position of the cell with respect to the central zone $O = r_0 \cap r_1 \cap \cdots \cap r_{k-1}$, which turns out to have label $(0, 0)$. In particular, the sign of x (resp. y) is positive if and only if the cell is on the right of (resp. above) O.

Finally, to each cell in the zone A (resp. B, C, and D) we associate the symbol a (resp. b, c, and d). Figure 10 shows the picture of Γ_4 which corresponds to the polyomino of Fig. 9 having four maximal rectangles.

The check that each picture representing an L-convex polyomino with k maximal rectangles belongs to $L(\theta^k)$ is immediate.

b) Lemma 3.3 assures that each set of cells P represented by a picture p of $L(\theta_k)$ is a convex polyomino. To prove that it is also L-convex, we choose any two elements of p having labels (x, y) and (x', y'), and we show the existence of a path connecting them, having at most one change of direction, and whose elements are labelled with couple of integers. So, the definition of θ^k allows us to achieve the L-convexity by simply proving that, for any two couples (x, y) and (x', y') in p, if $|x| + |y'| \geq k$ then $|x'| + |y| < k$. The constraints $|x| + |y| < k$, and $|x'| + |y'| < k$ directly lead to the thesis.

c) For any fixed k, we prove that no picture on Γ_k represents an L-convex polyomino having more than k maximal rectangles.

This result is achieved in two steps:

- *i*) we show that the value x (resp. y) of all the elements (x, y) in each row (resp. column) of a picture on Γ_k is constant, while the sequence of the values of the y (resp. x) weakly increases;
- *ii*) we use *i*) to show that two elements have different labels if and only if they belong to intersections of different sets of maximal rectangles.

The claim *i*) directly follows from the definition of the tiles of θ^k, i.e. in each tile, if the label (x_0, y_0) is above the label (x_1, y_1), immediately above it, then it holds $x_0 = x_1$, and $y_0 \geq y_1$. A symmetric result can be stated for the labels which lie in the same row. In the first case we call the number $x_0 = x_1$ *index* of their common column, while in the second we call $y_0 = y_1$ *index* of their common row.

The claim ii) needs a little bit more attention: by definition, each time two consecutive maximal rectangles r and r' of P intersect, there exists at least a 2×2 set of cells in the border of the polyomino which has one cell in $R(P) \setminus P$, i.e. outside the polyomino, one in $r \setminus r'$, one in $r' \setminus r$, and the fourth one in $r \cap r'$. Looking at the tiles of θ^k, each of these sets of cells is represented by one among the four tiles

$$\begin{array}{|c|c|}\hline a & (x, y+j) \\ \hline (x-i, y) & (x, y) \\ \hline \end{array}\,, \quad \begin{array}{|c|c|}\hline (x, y+j) & b \\ \hline (x, y) & (x+i, y) \\ \hline \end{array}\,,$$

$$\begin{array}{|c|c|}\hline (x-i, y) & (x, y) \\ \hline c & (x, y-j) \\ \hline \end{array}\,, \quad \begin{array}{|c|c|}\hline (x, y) & (x+i, y) \\ \hline (x, y-j) & d \\ \hline \end{array}\,,$$

with $i, j > 0$ (see Fig 10, (a)). So, one can easily check that moving along a single column (resp. a single row), each time we cross the border of a maximal rectangle, the row indexes (resp. the column indexes) change, until we reach the border of the polyomino.

Since for each label $(x, y) \in \Gamma_k$, we have $-k < x, y < k$, and using the result of the claim i), we obtain that the number of possible crossing of the borders of the maximal rectangles when moving along a single row or column is $2(k-1)$ at most, and consequently the number of maximal rectangles is k, as expected.

a	a	a	a	(0,3)	b	b	b
a	a	(−1,2)	(−1,2)	(0,2)	(1,2)	b	b
a	(−2,1)	(−1,1)	(−1,1)	(0,1)	(1,1)	(2,1)	(2,1)
a	(−2,1)	(−1,1)	(−1,1)	(0,1)	(1,1)	(2,1)	(2,1)
(−3,0)	(−2,0)	(−1,0)	(−1,0)	(0,0)	(1,0)	(2,0)	(2,0)
c	c	(−1,−1)	(−1,−1)	(0,−1)	(1,−1)	d	d
c	c	c	c	(0,−2)	d	d	d

Figure 10: The picture coding for the L-convex polyomino of Fig. 9. Two maximal rectangles are bordered and two of the four tiles used in the proof of Theorem 3.6, part c) are highlighted.

We would like to point out that Theorem 3.6 does not imply that the class $\mathcal{L}_{Conv}$ is tiling recognizable unless we admit that the tiling system can have an infinite alphabet $\Gamma = \bigcup_{k \geq 1} \Gamma_k$. Therefore the problem of establishing if the class of L-convex polyominoes is tiling recognizable is not solved yet.

However, as we observed in the introduction, the Theorem 3.6 is of particular interest since it shows the possibility of representing L-convexity, which is a global property of a polyomino, by means of a set of local adjacency constraints (i.e. the tiling system). A further effort should be made to check if other classes of polyominoes considered in literature, in particular those not defined by means of convexity constrains, can be represented by means of pictures of a tiling system.

4 On the nature of the generating functions of tiling systems

In this section we investigate some analytical properties of the generating functions of tiling systems.

Let us start by recalling some basic definitions from analysis [S]. A formal power series $u(x)$ with coefficients in $\mathbb{Q}$ is said to be *rational* if it can be written in the form:

$$u(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials in $\mathbb{Q}[x]$.

The series $u(x)$ is *algebraic* if there exist polynomials $p_0(x), \ldots, p_d(x) \in \mathbb{Q}[x]$, not all 0, such that:

$$p_0(x) + p_1(x)u + \ldots + p_d(x)u^d = 0.$$

Finally, $u(x)$ is said to be *differentiably finite* (briefly, *D-finite*), or *holonomic* if it satisfies a polynomial equation:

$$q_m(x)u^{(m)} + q_{m-1}(x)u^{(m-1)} + \ldots + q_1(x)u' + q_0(x)u = q(x),$$

with $q_0(x), \ldots, q_m(x) \in \mathbb{C}[x]$, and $q_m(x) \neq 0$.

These three classes of functions form a hierarchy of generating functions: rational generating functions are properly included in algebraic ones, and algebraic functions are included in the D-finite. For example, the generating function $u(x)$ of the sequence $\binom{2n}{n}$ is algebraic, since it satisfies the equation:

$$(1-4x)u^2 - 1 = 0,$$

while the generating function $v(x)$ of the sequence $\binom{2n}{n}^2$ is D-finite, since it satisfies the linear differential equation:

$$4v + (32x-1)v' + x(16x-1)v'' = 0,$$

but not algebraic, as proved in [F].

As we climb up the levels of the hierarchy, the generating functions become more and more complex, with regards to different combinatorial aspects (for instance, the asymptotic expansion, or the treatment using computer algebra). The generating functions of the most common solved models in mathematical physics are algebraic or differentiably finite, while models leading to non D-finite functions are usually considered "unsolvable" (see [BMP, BMR, Gu, R, Rc]).

To study the level in the hierarchy where to insert the generating functions of tiling systems, we still need to introduce some classical definitions and results from the theory of formal languages. The generating function $f_{\mathcal{L}}(x)$ of a language $\mathcal{L}$ is the formal power series $f_{\mathcal{L}}(x) = \sum_{n \geq 0} f_n x^n$, such that $\forall n \in \mathbb{N}$, $f_n = \|\{w \in \mathcal{L} : |w| = n\}\|$, where $|w|$ denotes the length of w. A classical result by Chomsky and Schützenberger [CS] states that:

Theorem 4.1

1. The generating function of a regular language is rational.

2. The generating function of an unambiguous context-free language is algebraic.

Passing to the two-dimensional case, we can naturally extend the notion of generating function of a language. So, let $\mathcal{L}$ be a two-dimensional language, the *bivariate generating function* of $\mathcal{L}$ is the formal power series:

$$f_{\mathcal{L}}(x,y) = \sum_{n,m \geq 1} f_{n,m} x^n y^m,$$

where $f_{n,m}$ is the number of pictures of size (n,m) in $\mathcal{L}$. More often we consider the *generating function* $f_{\mathcal{L}}(x)$ of $\mathcal{L}$:

$$f_{\mathcal{L}}(x) = f_{\mathcal{L}}(x, x) = \sum_{n \geq 1} f_n x^n.$$

Furthermore, given a tiling system $\mathcal{T} = (\Sigma, \Gamma, \theta, \varphi)$, the generating function of $\mathcal{T}$, denoted by $f_{\mathcal{T}}(x)$, is simply the generating function of the language $L(\mathcal{T})$ recognized by $\mathcal{T}$.

It is an open problem to determine an analogous of Theorem 4.1 for two-dimensional languages; in particular in this section we will study the problem of characterizing the set of generating functions of tiling systems.

We remark, as pointed out in several papers [GR], that tiling systems have been introduced as a two-dimensional extension of regular languages. In the rest of the section we will show, quite surprisingly, that the generating functions of tiling systems are not necessarily rational, but can also be algebraic, D-finite, and non D-finite. This fact certifies that tiling systems are capable of representing a large amount of combinatorial structures which cannot be handled using string languages.

The following remark points out one basic fact which will allow us to work with the generating functions of many tiling systems that we have considered in Section 2.

Remark 1 *Let $\mathcal{P}$ be a class of convex polyominoes, and let $\mathcal{T}$ be a tiling system recognizing the class $\mathcal{P}$. Then the generating function $f_{\mathcal{T}}(x)$ coincides with the generating function of the class $\mathcal{P}$ according to the semi-perimeter.*

Example 4.2 *Rational generating functions.* Many examples can be given of tiling systems having a rational generating function. Referring to the tiling systems we have presented in Section 2, we have that the generating function $f_{\mathcal{F}}(x)$ (resp. $f_{\mathcal{S}}(x)$) of the class $\mathcal{F}$ (resp. $\mathcal{S}$) is equal to the generating function of the class of Ferrers diagrams (resp. stack polyominoes) according to the semi-perimeter, which is well-known to be rational [S], and precisely:

$$f_{\mathcal{F}}(x) = \frac{x^2}{1-2x}, \qquad f_{\mathcal{S}}(x) = \frac{x^2(1-x)}{1-3x+x^2}.$$

Example 4.3 *Algebraic generating functions.* As in the previous case, there are various examples of tiling systems having an algebraic generating function. For instance, the generating function $f_{\mathcal{C}}(x)$ of the class $\mathcal{C}$ (of pictures representing convex polyominoes) is equal to the generating function of the class of convex polyominoes according to the semi-perimeter, which is algebraic [DV]:

$$f_{\mathcal{F}}(x) = x^2 \sum_{n \geq 0} f_n x^n = x^2 \left(\frac{1 - 6x + 11x^2 - 4x^3 - 4x^2\sqrt{1-4x}}{(1-4x)^2} \right).$$

Analogously, we have that the generating function of the classes $\mathcal{P}$, $\mathcal{D}$ are algebraic [BM].

Example 4.4 *D-finite generating functions.* We present a tiling system whose generating function is D-finite but not algebraic. Let us consider the class $\mathcal{D}^{\square}$ of pictures representing *square directed-convex polyominoes*, i.e. directed convex-polyominoes having the same number of rows and columns (see Fig. 11 (a)).

First we prove that $\mathcal{D}^{\square}$ is tiling recognizable. To do this we slightly modify the representation that we have used for the class $\mathcal{D}$ of pictures representing directed convex polyominoes, adding some more conditions with the aim of ensuring that the recognized picture is a square; in practice, to each square directed-convex polyomino having n rows and n columns, $n \geq 1$, we associate a picture of size (n, n) obtained by:

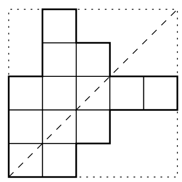
- representing with the symbol x (resp. y) every cell belonging (resp. not belonging) to the polyomino and lying on the diagonal from $(0, 0)$ to (n, n);

- representing with the symbol 1 every cell belonging to the polyomino, and not lying on the diagonal from $(0, 0)$ to (n, n);

- representing with the symbol a (resp. b, d) every cell in A (resp. B, D), and not lying on the diagonal from $(0, 0)$ to (n, n) (as depicted in Fig. 11 (b));

We remark that, by construction, the symbol y representing diagonal cells outside the polyomino can legitimately be in the zones A, B or D.

Let us denote by $\mathcal{L}_{\square}$ the language of such square pictures over the alphabet $\{1, a, b, d, x, y\}$. By simple computations the reader can easily prove that $\mathcal{L}_{\square}$ is a local language (for brevity we omit the set of tiles for $\mathcal{L}_{\square}$); furthermore if $\pi_{\square}$ is the projection such that $\pi_{\square}(1) = \pi_{\square}(x) = 1$, and $\pi_{\square}(a) = \pi_{\square}(b) = \pi_{\square}(d) = \pi_{\square}(y) = 0$, we have that $\pi_{\square}(\mathcal{L}_{\square}) = \mathcal{D}^{\square}$, thus $\mathcal{D}^{\square}$ is tiling recognizable.



(a)

#	#	#	#	#	#	#
#	*a*	*1*	*b*	*b*	*y*	#
#	*a*	*1*	*1*	*y*	*b*	#
#	*1*	*1*	*x*	*1*	*1*	#
#	*1*	*x*	*1*	*d*	*d*	#
#	*x*	*1*	*d*	*d*	*d*	#
#	#	#	#	#	#	#

(b)

Figure 11: (a) a square directed-convex polyomino; the diagonal from $(0,0)$ to (n,n) is traced; (b) its representation as a two-dimensional word in the alphabet $\{1,a,b,d,x,y\}$.

Finally, since [BM] the number of directed-convex polyominoes having n rows and n columns, $n \geq 1$, is equal to:

$$\binom{2n}{n}^2,$$

the generating function of such a sequence (hence the function $f_\Box(x)$) is D-finite but not algebraic.

Example 4.5 *Non D-finite generating functions.* In this final example we present a tiling system whose generating function is not D-finite.

We first need to recall some basics from the theory of *integer partitions*. Every integer partition (briefly, partition) $\lambda = (p_1, \ldots, p_l)$, with $p_1 \geq p_2 \geq \ldots \geq p_l \geq 1$, has a simple graphical representation in terms of *Ferrers diagrams* made of l rows such that the i-th column is made of p_i cells. A *partition into distinct parts* is a partition $\lambda = (p_1, \ldots, p_l)$, where $p_1 > p_2 > \ldots > p_l$; thus the associated Ferrers diagram is made of rows all having different lengths (see Fig. 12 (a)).

The generating function of partitions into distinct parts is well-known [S]:

$$d(x) = \prod_{k \geq 0} \frac{1}{1 - x^{2k+1}},$$

and it is not D-finite. In order to convince the reader of this fact, we can use the very simple argument, arising from the classical theory of linear differential equations (and applied firstly by Flajolet in [F]), that a D-finite power series of a single variable has only a finite number of singularities.

Thus, since $d(x)$ has infinitely many distinct poles (in fact, every $(2k+1)$th root of unity is a pole of $d(x)$, $k \geq 0$), it cannot be D-finite.

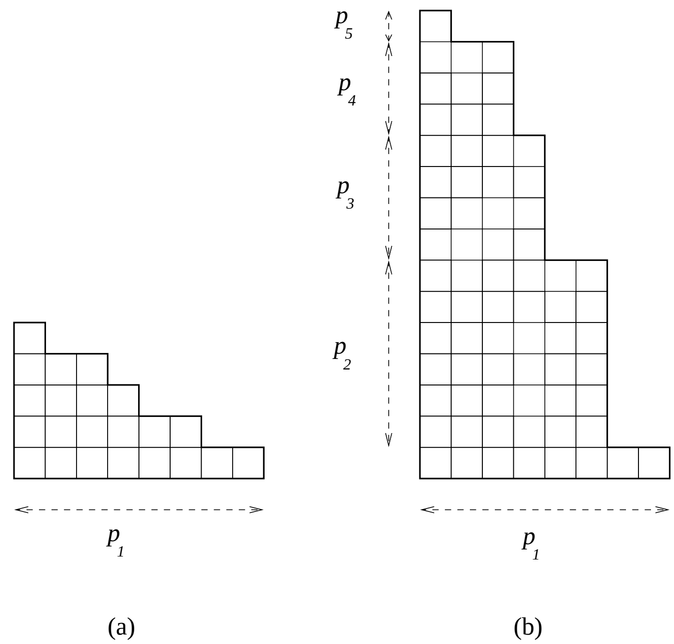


Figure 12: (a) the Ferrers diagram representing the partition into distinct parts $P = (8, 6, 4, 3, 1)$ of 22; (b) the polyomino $\psi(P)$ having semi-perimeter 23.

Our aim is to determine a class $\mathcal{Q}$ of polyominoes such that:

1. there exists a bijective function ψ:

$$\psi : \mathcal{W}_n \rightarrow \mathcal{Q}_n,$$

   where, for $n \geq 1$, $\mathcal{W}_n$ is the class of Ferrers diagrams associated with partitions of n into distinct parts, and $\mathcal{Q}_n$ is the set of polyominoes of $\mathcal{Q}$ with semi-perimeter $n + 1$.

2. the language $\overline{\mathcal{Q}}$ of pictures representing the polyominoes of $\mathcal{Q}$ is tiling recognizable.

In practice, the function ψ transforms a Ferrers diagram of area n into a Ferrers diagram of semi-perimeter $n + 1$.

1. We reach our goal by defining an injective function ψ from $\mathcal{W}$ to the whole set of Ferrers diagrams, and then setting $\mathcal{Q} = \psi\,[\mathcal{W}]$.

   For the sake of simplicity we will describe the bijection without going deep into formalisms. Let $P \in \mathcal{W}$ and let $r_1, \ldots, r_k$ be the rows of P; for $i = 1, \ldots, k$ let $p_i \geq 1$ be the number of cells of r_i; then $\psi(P)$ is a Ferrers diagram obtained as follows:

   i) the first row R_1 of $\psi(P)$ is made of p_1 cells;

   ii) for each $i \geq 2$ we place a square R_i of side p_i, just above R_{i-1}, and such that its leftmost column is placed just on the top of the leftmost cell of the upper row of R_{i-1} (see Fig. 12 (b)).

   By construction, we have that the semi-perimeter of $\psi(P)$ is given by $p_1 + 1 + p_2 + \ldots + p_k$, and this sum is equal to the area of P plus one, hence ψ satisfies Property 1.

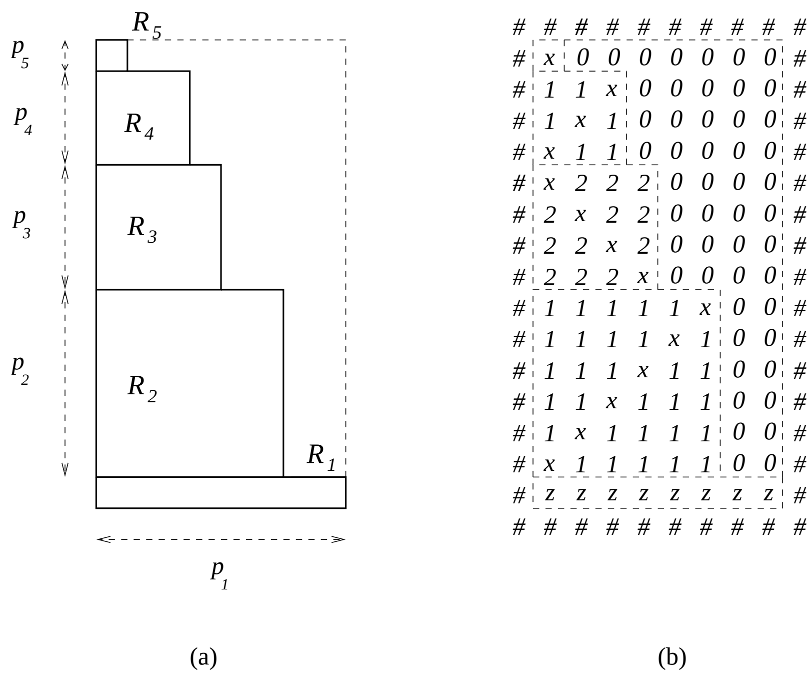


Figure 13: (a) the *square decomposition* of $\psi(P)$, where P is the polyomino depicted in Fig. 12 (b); (b) the representation of $\psi(P)$ as a word of $\mathcal{L}_Q$ in the alphabet $\{0, 1, 2, z, x\}$.

2. Thus, let $\overline{\mathcal{Q}}$ be the language of pictures representing the polyominoes of $\mathcal{Q}$. In order to prove that $\overline{\mathcal{Q}}$ is tiling recognizable we use the same technique applied in the previous examples: we consider a coding of the pictures of $\overline{\mathcal{Q}}$ as pictures of a local language $\mathcal{L}_Q$ over the alphabet $\{0, 1, 2, x, z\}$.

   We recall that $Q \in \overline{\mathcal{Q}}$ can be decomposed, as suggested in Fig. 13 (a), into $k-1$ disjoint squares $R_2, \dots, R_k$, plus the first row R_1 made of p_1 cells. The coding of Q is obtained as follows:

   (a) the elements in the row R_1 are encoded by the symbol z;

   (b) the elements in the square R_{2i}, $i \geq 1$ are encoded by the symbol 1, except those lying on the diagonal going from the bottom on the left to the top on the right, which are encoded by the symbol x;

   (c) the elements in the square R_{2i+1}, $i \geq 1$ are encoded by the symbol 2, except those lying on the diagonal going from the bottom on the right to the top on the left, which are encoded by the symbol x.

   Figure 13 (b) shows the representation of $\psi(P)$ as a word of $\mathcal{L}_Q$, where P is the polyomino depicted in Fig. 12 (b). It is not difficult to prove that $\mathcal{L}_Q$ is a local language, and letting π_Q be defined as: $\pi_Q(1) = \pi_Q(2) = \pi_Q(x) = \pi_Q(z) = 1$, and $\pi_Q(0) = 0$, we have that $\pi_Q(\mathcal{L}_Q) = \overline{\mathcal{Q}}$; henceforth $\overline{\mathcal{Q}}$ is tiling recognizable.

   Since the generating function of $\overline{\mathcal{Q}}$ is equal to the generating function of $\mathcal{Q}$ according to the semi-perimeter, and this latter function is equal to the generating function of partitions into distinct parts according to the area multiplied by x^2, the final corollary is straightforward.

   Corollary 4.6 *The generating function of $\overline{\mathcal{Q}}$ is equal to:*

$$x^2\, d(x) = x^2 \prod_{k \geq 0} \frac{1}{1 - x^{2k+1}},$$

   hence it is not D-finite.

This final section leaves many interesting open problems to the reader, which require a deep analytical study of the properties of the generating functions of tiling systems; in particular: is there a methodology (analogous to the Schützenberger methodology) to pass from a tiling system (or at least some special classes of tiling system) to its generating function? are there classes of tiling systems for which an analogous of Theorem 4.1 holds? are there classes leading to rational generating functions?

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